CHAPTER 9

Applications in Solid Mechanics

9.1 INTRODUCTION

The bar and beam elements discussed in Chapters 2–4 are line elements, as only a single coordinate axis is required to define the element reference frame, hence, the stiffness matrices. As shown, these elements can be successfully used to model truss and frame structures in two and three dimensions. For application of the finite element method to more general solid structures, the line elements are of little use, however. Instead, elements are needed that can be used to model complex geometries subjected to various types of loading and constraint conditions.

In this chapter, we develop the finite element equations for both two- and three-dimensional elements for use in stress analysis of linearly elastic solids. The principle of minimum potential energy is used for the developments, as that principle is somewhat easier to apply to solid mechanics problems than Galerkin's method. It must be emphasized, however, that Galerkin's method is the more general procedure and applicable to a wider range of problems.

The constant strain triangle for plane stress is considered first, as the CST is the simplest element to develop mathematically. The procedure is shown to be common to other elements as well; a rectangular element formulated for plane strain is used to illustrate this commonality. Plane quadrilateral, axisymmetric, and general three-dimensional elements are also examined. An approach for application of the finite element method to solving torsion problems of noncircular sections is also presented.

9.2 PLANE STRESS

A commonly occurring situation in solid mechanics, known as *plane stress,* is defined by the following assumptions in conjunction with Figure 9.1:

1. The body is small in one coordinate direction (the z direction by convention) in comparison to the other dimensions; the dimension in the z direction (hereafter, the thickness) is either uniform or symmetric about the xy plane; thickness t, if in general, is less than one-tenth of the smallest dimension in the xy plane, would qualify for "small."
2. The body is subjected to loading only in the xy plane.
3. The material of the body is linearly elastic, isotropic, and homogeneous.

The last assumption is not required for plane stress but is utilized in this text as we consider only elastic deformations.

Given a situation that satisfies the plane stress assumptions, the only nonzero stress components are σ_x, σ_y, and τ_{xy}. Note that the nominal stresses perpendicular to the xy plane (σ_z, τ_{xz}, τ_{yz}) are zero as a result of the plane stress assumptions. Therefore, the equilibrium equations (Appendix B) for plane stress are

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial x} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \end{aligned} \tag{9.1}$$

where we implicitly assume that $\tau_{xy} = \tau_{yx}$. Utilizing the elastic stress-strain relations from Appendix B, Equation B.12 with $\sigma_z = \tau_{xz} = \tau_{yz} = 0$, the nonzero stress components can be expressed as (Problem 9.1)

$$\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \\ \sigma_y &= \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \\ \tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy} = G\gamma_{xy} \end{aligned} \tag{9.2}$$

where E is the modulus of elasticity and ν is Poisson's ratio for the material. In the shear stress-strain relation, the shear modulus $G = E/2(1+\nu)$ has been introduced.

The stress-strain relations given by Equation 9.2 can be conveniently written in matrix form:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{9.3}$$

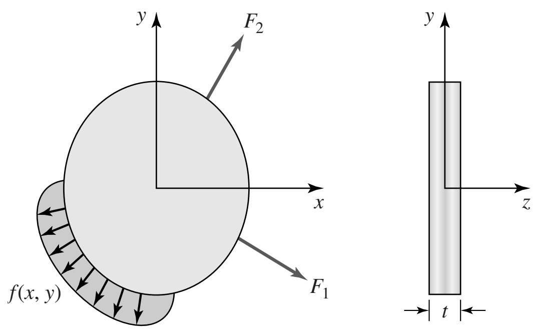


Figure 9.1 An illustration of plane stress conditions.

or

$$\{\sigma\} = [D]\{\varepsilon\} \tag{9.4}$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{9.5}$$

is the column matrix of stress components,

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{9.6}$$

is the elastic material property matrix for plane stress, and

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{9.7}$$

is the column matrix of strain components.

For a state of plane stress, the strain energy per unit volume, Equation 2.43, becomes

$$u_e = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy}) \tag{9.8}$$

or, using the matrix notation,

$$u_e = \frac{1}{2}\{\varepsilon\}^T\{\sigma\} = \frac{1}{2}\{\varepsilon\}^T[D]\{\varepsilon\} \tag{9.9}$$

Use of $\{\varepsilon\}^T$ allows the matrix operation to reproduce the quadratic form of the strain energy. Note that a quadratic relation in any variable z can be expressed as $\{z\}^T[A]\{z\}$, where $[A]$ is a coefficient matrix. This is the subject of an end-of-chapter problem.

The total strain energy of a body subjected to plane stress is then

$$U_e = \frac{1}{2}\iiint\limits_V \{\varepsilon\}^T[D]\{\varepsilon\}\,\mathrm{d}V \tag{9.10}$$

where V is total volume of the body and $\mathrm{d}V = t\,\mathrm{d}x\,\mathrm{d}y$. The form of Equation 9.10 will in fact be found to apply in general and is not restricted to the case of plane stress. In other situations, the strain components and material property matrix may be defined differently, but the form of the strain energy expression does not change. We use this result extensively in applying the principle of minimum potential energy in following developments.

9.2.1 Finite Element Formulation: Constant Strain Triangle

Figure 9.2a depicts a three-node triangular element assumed to represent a subdomain of a body subjected to plane stress. Element nodes are numbered as shown, and nodal displacements in the x-coordinate direction are u_1, u_2, and u_3, while displacements in the y direction are v_1, v_2, and v_3. (For plane stress, displacement in the z direction is neglected). As noted in the introduction, the displacement field in structural problems is a vector field and must be discretized accordingly. For the triangular element in plane stress, we write the discretized displacement field as

$$\begin{aligned} u(x, y) &= N_1(x, y)u_1 + N_2(x, y)u_2 + N_3(x, y)u_3 = [N]\{u\} \\ v(x, y) &= N_1(x, y)v_1 + N_2(x, y)v_2 + N_3(x, y)v_3 = [N]\{v\} \end{aligned} \tag{9.11}$$

where N_1, N_2, and N_3 are the interpolation functions as defined in Equation 6.37. Using the discretized representation of the displacement field, the element strain

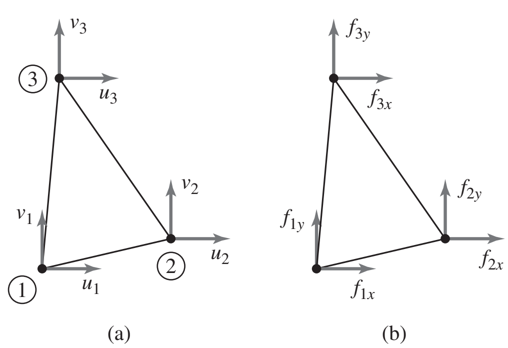


Figure 9.2
(a) Nodal displacement notation for a plane stress element. (b) Nodal forces.

components are then

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x}u_1 + \frac{\partial N_2}{\partial x}u_2 + \frac{\partial N_3}{\partial x}u_3 \\
\varepsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y}v_1 + \frac{\partial N_2}{\partial y}v_2 + \frac{\partial N_3}{\partial y}v_3 \qquad (9.12) \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial N_1}{\partial y}u_1 + \frac{\partial N_2}{\partial y}u_2 + \frac{\partial N_3}{\partial y}u_3 + \frac{\partial N_1}{\partial x}v_1 + \frac{\partial N_2}{\partial x}v_2 + \frac{\partial N_3}{\partial x}v_3
\end{aligned}
$$

Defining the element displacement column matrix (vector) as

$$
\left\{\delta^{(e)}\right\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} \tag{9.13}
$$

the element strain matrix can be expressed as

$$
\{\varepsilon\} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = [B]\left\{\delta^{(e)}\right\} \tag{9.14}
$$

where $[B]$ is the 3×6 matrix of partial derivatives of the interpolation functions as indicated, also known as the strain-displacement matrix. Referring to Equation 6.37, we observe that the partial derivatives appearing in Equation 9.14 are constants, since the interpolation functions are linear in the spatial variables. Hence, the strain components are constant throughout the volume of the element. Consequently, the three-node, triangular element for plane stress is known as a *constant strain triangle*.

By direct analogy with Equation 9.10, the elastic strain energy of the element is

$$
U_e^{(e)} = \frac{1}{2}\iiint\limits_{V^{(e)}} \{\varepsilon\}^T[D]\{\varepsilon\}\,\mathrm{d}V^{(e)} = \frac{1}{2}\iiint\limits_{V^{(e)}} \left\{\delta^{(e)}\right\}^T[B]^T[D][B]\left\{\delta^{(e)}\right\}\mathrm{d}V^{(e)} \tag{9.15}
$$

As shall be seen in subsequent examples, Equation 9.15 is a generally applicable relation for the elastic strain energy of structural elements. For the constant strain triangle, we already observed that the strains are constant over the element volume. Assuming that the elastic properties similarly do not vary, Equation 9.15

becomes simply

$$U_e^{(e)} = \frac{1}{2}\{\delta^{(e)}\}^T[B]^T[D][B]\{\delta^{(e)}\}\iiint_{V^{(e)}} \mathrm{d}V^{(e)}$$
$$= \frac{1}{2}\{\delta^{(e)}\}^T\left(V^{(e)}[B]^T[D][B]\right)\{\delta^{(e)}\} \tag{9.16}$$

where $V^{(e)}$ is the total volume of the element.

Considering the element forces to be as in Figure 9.2b (for this element formulation, we require that forces be applied only at nodes; distributed loads are considered subsequently), the work done by the applied forces can be expressed as

$$W = f_{1x}u_1 + f_{2x}u_2 + f_{3x}u_3 + f_{1y}v_1 + f_{2y}v_2 + f_{3y}v_3 \tag{9.17}$$

and we note that the subscript notation becomes unwieldy rather quickly in the case of 2-D stress analysis. To simplify the notation, we use the force notation

$$\{f\} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{3x} \\ f_{1y} \\ f_{2y} \\ f_{3y} \end{Bmatrix} \tag{9.18}$$

so that we can express the work of the external forces (using Equation 9.13) as

$$W = \{\delta\}^T\{f\} \tag{9.19}$$

Per Equation 2.53, the total potential energy for an element is then

$$\Pi = U_e - W = \frac{V^e}{2}\{\delta\}^T[B]^T[D][B]\{\delta\} - \{\delta\}^T\{f\} \tag{9.20}$$

If the element is a portion of a larger structure that is in equilibrium, then the element must be in equilibrium. Consequently, the total potential energy of the element must be minimum (we consider only stable equilibrium), and for this minimum, we must have mathematically

$$\frac{\partial\Pi}{\partial\delta_i} = 0 \qquad i = 1, 6 \tag{9.21}$$

If the indicated mathematical operations of Equation 9.21 are carried out on Equation 9.20, the result is the matrix relation

$$V^e[B]^T[D][B]\{\delta\} = \{f\} \tag{9.22}$$

and this matrix equation is of the form

$$[k]\{\delta\} = \{f\} \tag{9.23}$$

where $[k]$ is the element stiffness matrix defined by

$$[k] = V^e[B]^T[D][B] \tag{9.24}$$

and we must keep in mind that we are dealing with only a constant strain triangle at this point.

This theoretical development may not be obvious to the reader. To make the process more clear, especially the application of Equation 9.21, we examine the element stiffness matrix in more detail. First, we represent Equation 9.20 as

$$\Pi = \frac{1}{2}\{\delta\}^T[k]\{\delta\} - \{\delta\}^T\{f\} \tag{9.25}$$

and expand the relation formally to obtain the *quadratic* function

$$\begin{aligned}\Pi = \frac{1}{2}\big(&k_{11}\delta_1^2 + 2k_{12}\delta_1\delta_2 + 2k_{13}\delta_1\delta_3 + 2k_{14}\delta_1\delta_4 + \cdots + 2k_{56}\delta_5\delta_6 + k_{66}\delta_6^2\big)\\ &- f_{1x}\delta_1 - f_{2x}\delta_2 - f_{3x}\delta_3 - f_{1y}\delta_4 - f_{2y}\delta_5 - f_{3y}\delta_6\end{aligned} \tag{9.26}$$

The quadratic function representation of total potential energy is characteristic of linearly elastic systems. (Recall the energy expressions for the strain energy of spring and bar elements of Chapter 2.)

The partial derivatives of Equation 9.21 are then in the form

$$\begin{aligned}\frac{\partial \Pi}{\partial \delta_1} &= k_{11}\delta_1 + k_{12}\delta_2 + k_{13}\delta_3 + k_{14}\delta_4 + k_{15}\delta_5 + k_{16}\delta_6 - f_{1x} = 0\\ \frac{\partial \Pi}{\partial \delta_2} &= k_{21}\delta_1 + k_{22}\delta_2 + k_{23}\delta_3 + k_{24}\delta_4 + k_{25}\delta_5 + k_{26}\delta_6 - f_{2x} = 0\end{aligned} \tag{9.27}$$

for example. Equations 9.27 are the scalar equations representing equilibrium of nodes 1 and 2 in the *x*-coordinate direction. The remaining four equations similarly represent nodal equilibrium conditions in the respective coordinate directions.

As we are dealing with an elastic element, the stiffness matrix should be symmetric. Examining Equation 9.27, we should have $k_{12} = k_{21}$, for example. Whether this is the case may not be obvious in consideration of Equation 9.24, since $[D]$ is a symmetric matrix but $[B]$ is not symmetric. A fundamental property of matrix multiplication (Appendix A) is as follows: If $[G]$ is a real, symmetric $N \times N$ matrix and $[F]$ is a real $N \times M$ matrix, the matrix triple product $[F]^T[G][F]$ is a real, symmetric $M \times M$ matrix. Thus, the stiffness matrix as given by Equation 9.24 is a symmetric 6×6 matrix, since $[D]$ is 3×3 and symmetric and $[B]$ is a 6×3 real matrix.

9.2.2 Stiffness Matrix Evaluation

The stiffness matrix for the constant strain triangle element given by Equation 9.24 is now evaluated in detail. The interpolation functions per

Equation 6.37 are

$$\begin{aligned} N_1(x, y) &= \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] \\ &= \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y) \\ N_2(x, y) &= \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] \\ &= \frac{1}{2A}(\alpha_2 + \beta_2 x + \gamma_2 y) \\ N_3(x, y) &= \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] \\ &= \frac{1}{2A}(\alpha_3 + \beta 2x + \gamma_3 y) \end{aligned} \tag{9.28}$$

so the required partial derivatives are

$$\begin{aligned} &\frac{\partial N_1}{\partial x} = \frac{1}{2A}(y_2 - y_3) = \frac{\beta_1}{2A} \\ &\frac{\partial N_2}{\partial x} = \frac{1}{2A}(y_3 - y_1) = \frac{\beta_2}{2A} \qquad \frac{\partial N_3}{\partial x} = \frac{1}{2A}(y_1 - y_2) = \frac{\beta_3}{2A} \\ &\frac{\partial N_1}{\partial y} = \frac{1}{2A}(x_3 - x_2) = \frac{\gamma_1}{2A} \\ &\frac{\partial N_2}{\partial y} = \frac{1}{2A}(x_1 - x_3) = \frac{\gamma_2}{2A} \qquad \frac{\partial N_3}{\partial y} = \frac{1}{2A}(x_2 - x_1) = \frac{\gamma_3}{2A} \end{aligned} \tag{9.29}$$

The $[B]$ (strain-displacement) matrix is then

$$\begin{aligned} [B] &= \frac{1}{2A}\begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 & y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \end{bmatrix} \\ &= \frac{1}{2A}\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \end{aligned} \tag{9.30}$$

Noting that, for constant thickness, element volume is tA, substitution into Equation 9.24 results in

$$[k] = \frac{Et}{4A(1-\nu^2)}\begin{bmatrix} \beta_1 & 0 & \gamma_1 \\ \beta_2 & 0 & \gamma_2 \\ \beta_3 & 0 & \gamma_3 \\ 0 & \gamma_1 & \beta_1 \\ 0 & \gamma_2 & \beta_2 \\ 0 & \gamma_3 & \beta_3 \end{bmatrix}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \tag{9.31}$$

Performing the matrix multiplications of Equation 9.31 gives the element stiffness matrix as

$$[k] = \frac{Et}{4A(1-\nu^2)}\begin{bmatrix} \beta_1^2 + C\gamma_1^2 & \beta_1\beta_2 + C\gamma_1\gamma_2 & \beta_1\beta_3 + C\gamma_1\gamma_3 & \frac{1+\nu}{2}\beta_1\gamma_1 & \nu\beta_1\gamma_2 + C\beta_2\gamma_1 & \nu\beta_1\gamma_3 + C\beta_3\gamma_1 \\ & \beta_2^2 + C\gamma_2^2 & \beta_2\beta_3 + C\gamma_2\gamma_3 & \nu\beta_2\gamma_1 + C\beta_1\gamma_2 & \frac{1+\nu}{2}\beta_2\gamma_2 & \nu\beta_2\gamma_3 + C\beta_3\gamma_2 \\ & & \beta_3^2 + C\gamma_3^2 & \nu\beta_3\gamma_1 + C\beta_1\gamma_3 & \nu\beta_3\gamma_2 + C\beta_2\gamma_3 & \frac{1+\nu}{2}\beta_3\gamma_3 \\ & SYM & & \gamma_1^2 + C\beta_1^2 & \gamma_1\gamma_2 + C\beta_1\beta_2 & \gamma_1\gamma_3 + C\beta_1\beta_3 \\ & & & & \gamma_2^2 + C\beta_2^2 & \gamma_2\gamma_3 + C\beta_2\beta_3 \\ & & & & & \gamma_3^2 + C\beta_3^2 \end{bmatrix} \tag{9.32}$$

where $C = (1-\nu)/2$. Equation 9.32 is the explicit representation of the stiffness matrix for a constant strain triangular element in plane stress, presented for illustrative purposes. In finite element software, such explicit representation is not often used; instead, the matrix triple product of Equation 9.24 is applied directly to obtain the stiffness matrix.

9.2.3 Distributed Loads and Body Force

Frequently, the boundary conditions for structural problems involve distributed loading on some portion of the geometric boundary. Such loadings may arise from applied pressure (normal stress) or shearing loads. In plane stress, these distributed loads act on element edges that lie on the global boundary. As a general example, Figure 9.3a depicts a CST element having normal and tangential loads p_n and p_t acting along the edge defined by element nodes 2 and 3. Element thickness is denoted t, and the loads are assumed to be expressed in terms of force per unit area. We seek to replace the distributed loads with equivalent forces acting at nodes 2 and 3. In keeping with the minimum potential energy approach, the concentrated nodal loads are determined such that the mechanical work is the same as that of the distributed loads.

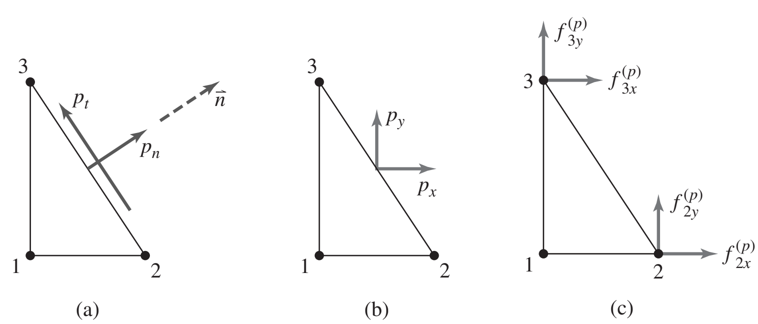


Figure 9.3 Conversion of distributed loading to work-equivalent nodal forces.

First, the distributed loads are converted to equivalent loadings in the global coordinate directions, as in Figure 9.3b, via

$$\begin{aligned} p_x &= p_n n_x - p_t n_y \\ p_y &= p_n n_y + p_t n_x \end{aligned} \tag{9.33}$$

with n_x and n_y corresponding to the components of the unit outward normal vector to edge 2-3. Here, we use the notation p for such loadings, as the units are those of pressure. The mechanical work done by the distributed loads is

$$W_p = t\int_2^3 p_x u(x, y)\,\mathrm{d}S + t\int_2^3 p_y v(x, y)\,\mathrm{d}S \tag{9.34}$$

where the integrations are performed along the edge defined by nodes 2 and 3. Recalling that interpolation function $N_1(x, y)$ is zero along edge 2-3, the finite element representations of the displacements along the edge are

$$\begin{aligned} u(x, y) &= N_2(x, y)u_2 + N_3(x, y)u_3 \\ v(x, y) &= N_2(x, y)v_2 + N_3(x, y)v_3 \end{aligned} \tag{9.35}$$

The work expression becomes

$$\begin{aligned} W_p = t\int_2^3 & p_x[N_2(x, y)u_2 + N_3(x, y)u_3]\,\mathrm{d}S \\ & + t\int_2^3 p_y[N_2(x, y)v_2 + N_3(x, y)v_3]\,\mathrm{d}S \end{aligned} \tag{9.36}$$

and is of the form

$$W_p = f_{2x}^{(p)}u_2 + f_{3x}^{(p)}u_3 + f_{2y}^{(p)}v_2 + f_{3y}^{(p)}v_3 \tag{9.37}$$

Comparison of the last two equations yields the equivalent nodal forces as

$$\begin{aligned} f_{2x}^{(p)} &= t\int_2^3 p_x N_2(x, y)\,\mathrm{d}S \\ f_{3x}^{(p)} &= t\int_2^3 p_x N_3(x, y)\,\mathrm{d}S \\ f_{2y}^{(p)} &= t\int_2^3 p_y N_2(x, y)\,\mathrm{d}S \\ f_{3x}^{(p)} &= t\int_2^3 p_y N_3(x, y)\,\mathrm{d}S \end{aligned} \tag{9.38}$$

as depicted in Figure 9.3c. Recalling again for emphasis that $N_1(x, y)$ is zero along the integration path, Equation 9.38 can be expressed in the compact form

$$\left\{f^{(p)}\right\} = \int_S [N]^T \left\{\begin{matrix} p_x \\ p_y \end{matrix}\right\} t \, \mathrm{d}S \tag{9.39}$$

with

$$[N]^T = \begin{bmatrix} N_1 & 0 \\ N_2 & 0 \\ N_3 & 0 \\ 0 & N_1 \\ 0 & N_2 \\ 0 & N_3 \end{bmatrix} \tag{9.40}$$

$$\left\{f^{(p)}\right\} = \begin{Bmatrix} f_{1x}^{(p)} \\ f_{2x}^{(p)} \\ f_{3x}^{(p)} \\ f_{1y}^{(p)} \\ f_{2y}^{(p)} \\ f_{3y}^{(p)} \end{Bmatrix} \tag{9.41}$$

The reader is urged to write out in detail the matrix multiplication indicated in Equation 9.39 to ensure that the result is correct. Although developed in the context of the three-node triangular element, Equation 9.39 will prove generally applicable for two-dimensional elements and require only minor modification for application to three-dimensional problems.

EXAMPLE 9.1

Given the triangular plane stress element shown in Figure 9.4a, determine the nodal forces equivalent to the distributed loads shown via the method of work equivalence discussed previously. Element thickness is 0.2 in. and uniform.

■ **Solution**

Using the nodal coordinates specified, the interpolation functions (with element area $A = 1$) are

$$N_1(x, y) = \frac{1}{2}[2 - 2x] = 1 - x$$

$$N_2(x, y) = \frac{1}{2}[2x - y]$$

$$N_3(x, y) = \frac{1}{2}y$$

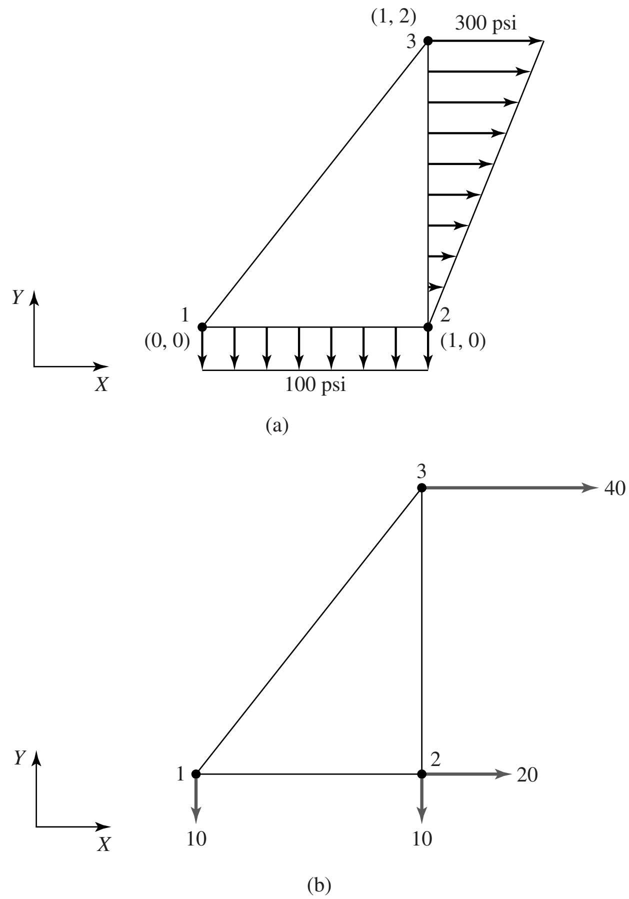


Figure 9.4
(a) Distributed loads on a triangular element.
(b) Work-equivalent nodal forces.

Along edge 1-2, $y = 0$, $p_x = 0$, $p_y = -100$ psi; hence, Equation 9.39 becomes

$$\{f^{(p)}\} = \int_S \begin{bmatrix} 1-x & 0 \\ x & 0 \\ 0 & 0 \\ 0 & 1-x \\ 0 & x \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ -100 \end{Bmatrix} t \, \mathrm{d}S$$

$$= 0.2 \int_0^1 \begin{bmatrix} 1-x & 0 \\ x & 0 \\ 0 & 0 \\ 0 & 1-x \\ 0 & x \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ -100 \end{Bmatrix} \mathrm{d}x = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -10 \\ -10 \\ 0 \end{Bmatrix} \text{lb}$$

For edge 2-3, we have $x = 1$, $p_x = 150y$, $p_y = 0$, so that

$$\{f^{(p)}\} = \int_S \begin{bmatrix} 0 & 0 \\ \frac{1}{2}(2-y) & 0 \\ \frac{y}{2} & 0 \\ 0 & \frac{1}{2}(2-y) \\ 0 & \frac{y}{2} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 150y \\ 0 \end{Bmatrix} t\,\mathrm{d}S$$

$$= 0.2 \int_0^2 \begin{bmatrix} 0 & 0 \\ \frac{1}{2}(2-y) & 0 \\ \frac{y}{2} & 0 \\ 0 & \frac{1}{2}(2-y) \\ 0 & \frac{y}{2} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 150y \\ 0 \end{Bmatrix} \mathrm{d}y = \begin{Bmatrix} 0 \\ 20 \\ 40 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \text{lb}$$

Combining the results, the nodal force vector arising from the distributed loads for the element shown is then

$$\{f^{(p)}\} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{3x} \\ f_{1y} \\ f_{2y} \\ f_{3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \\ 40 \\ -10 \\ -10 \\ 0 \end{Bmatrix} \text{lb}$$

as shown in Figure 9.4b.

In addition to distributed edge loads on element boundaries, so-called body forces may also arise. In general, a body force is a noncontact force acting on a body on a per unit mass basis. The most commonly encountered body forces are gravitational attraction (weight), centrifugal force arising from rotational motion, and magnetic force. Currently, we consider only the two-dimensional case in which the body force is described by the vector $\begin{Bmatrix} F_{BX} \\ F_{BY} \end{Bmatrix}$ in which F_{BX} and F_{BY} are forces per unit mass acting on the body in the respective coordinate directions. As with distributed loads, the body forces are to be replaced by equivalent nodal forces. Considering a differential mass $\rho t\,\mathrm{d}x\,\mathrm{d}y$ undergoing displacements (u, v) in the coordinate directions, mechanical work done by the body forces is

$$\mathrm{d}W_b = \rho F_{BX} ut \,\mathrm{d}x\,\mathrm{d}y + \rho F_{BY} vt \,\mathrm{d}x\,\mathrm{d}y \tag{9.42}$$

Considering the volume of interest to be a CST element in which the displacements are expressed in terms of interpolation functions and nodal displacements as

$$\begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = [N] \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{9.43}$$

the total work done by the body forces acting on the element is expressed in terms of nodal displacement as

$$\begin{aligned} W_b = \rho t \iint\limits_A F_{BX}(N_1 u_1 + N_2 u_2 + N_3 u_3)\,\mathrm{d}x\,\mathrm{d}y \\ + \rho t \iint\limits_A F_{BY}(N_1 v_1 + N_2 v_2 + N_3 v_3)\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{9.44}$$

As desired, Equation 9.44 is in the form

$$W_b = f_{1x}^{(b)} u_1 + f_{2x}^{(b)} u_2 + f_{3x}^{(b)} u_3 + f_{1y}^{(b)} v_1 + f_{2y}^{(b)} v_2 + f_{3y}^{(b)} v_3 \tag{9.45}$$

in terms of equivalent concentrated nodal forces. The superscript (b) is used to indicate nodal-equivalent body force. Comparison of the last two equations yields the nodal force components as

$$\begin{aligned} f_{ix}^{(b)} &= \rho t \int\limits_A N_i F_{BX}\,\mathrm{d}x\,\mathrm{d}y \qquad i = 1, 3 \\ f_{iy}^{(b)} &= \rho t \int\limits_A N_i F_{BY}\,\mathrm{d}x\,\mathrm{d}y \qquad i = 1, 3 \end{aligned} \tag{9.46}$$

The nodal force components equivalent to the applied body forces can also be written in the compact matrix form

$$\left\{ f^{(b)} \right\} = \rho t \int\limits_A [N]^T \begin{Bmatrix} F_{BX} \\ F_{BY} \end{Bmatrix} \mathrm{d}x\,\mathrm{d}y \tag{9.47}$$

While developed in the specific context of a constant strain triangular element in plane stress, Equation 9.47 proves to be a general result for two-dimensional elements. A quite similar expression holds for three-dimensional elements.

EXAMPLE 9.2

Determine the nodal force components representing the body force for the element of Example 9.1, if the body force is gravitational attraction in the y direction, so that

$$\begin{Bmatrix} F_{BX} \\ F_{BY} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -386.4 \end{Bmatrix} \text{ in./sec}^2$$

given the density of the element material is $\rho = 7.3 \times 10^{-4}$ slug/in.^{3}.

■ Solution

As the x component of the body force is zero, the x components of the nodal force vector will be, too, so we need not consider those components. The y components are computed using the second of Equation 9.46:

$$f_{iy}^{(b)} = \rho t \int_A N_i F_{BY} \, \mathrm{d}x \, \mathrm{d}y \qquad i = 1, 3$$

From the previous example, the interpolation functions are

$$N_1(x, y) = \frac{1}{2}[2 - 2x] = 1 - x$$

$$N_2(x, y) = \frac{1}{2}[2x - y]$$

$$N_3(x, y) = \frac{1}{2}y$$

We have, in this instance,

$$f_{1y}^{(b)} = \rho t \iint_A F_{BY} N_1 \, \mathrm{d}x \, \mathrm{d}y = \rho t \iint_A F_{BY}(1 - x) \, \mathrm{d}x \, \mathrm{d}y$$

$$f_{2y}^{(b)} = \rho t \iint_A F_{BY} N_2 \, \mathrm{d}x \, \mathrm{d}y = \rho t \iint_A \frac{F_{BY}}{2}(2x - y) \, \mathrm{d}x \, \mathrm{d}y$$

$$f_{3y}^{(b)} = \rho t \iint_A F_{BY} N_3 \, \mathrm{d}x \, \mathrm{d}y = \rho t \iint_A \frac{F_{BY}}{2} y \, \mathrm{d}x \, \mathrm{d}y$$

The limits of integration must be determined on the basis of the geometry of the area. In this example, we utilize x as the basic integration variable and compute the y-integration limits in terms of x. For the element under consideration, as x varies between zero and one, y is the linear function $y = 2x$ so the integrations become

$$f_{1y}^{(b)} = \rho t F_{BY} \int_0^1 \int_0^{2x} (1 - x) \, \mathrm{d}y \, \mathrm{d}x = \rho t F_{BY} \int_0^1 2x(1 - x) \, \mathrm{d}x = \rho t Y \left(x^2 - \frac{2x^3}{3} \right) \Bigg|_0^1$$

$$= \frac{\rho t F_{BY}}{3} = -0.0189 \text{ lb}$$

$$f_{2y}^{(b)} = \rho t F_{BY} \int_0^1 \int_0^{2x} \frac{1}{2}(2x - y)\,\mathrm{d}y\,\mathrm{d}x = \frac{\rho t F_{BY}}{2} \int_0^1 2x^2\,\mathrm{d}x$$

$$= \frac{\rho t F_{BY}}{2}\left(\frac{2}{3}\right) = \frac{\rho t F_{BY}}{3} = -0.0189 \text{ lb}$$

$$f_{3y}^{(b)} = \frac{\rho t F_{BY}}{2} \int_0^1 \int_0^{2x} y\,\mathrm{d}x\,\mathrm{d}y = \frac{\rho t F_{BY}}{2} \int_0^1 2x^2\,\mathrm{d}x = \frac{\rho t F_{BY}}{3} = -0.0189 \text{ lb}$$

showing that the body force is equally distributed to the element nodes.

If we now combine the concepts just developed for the CST element in plane stress, we have a general element equation that includes directly applied nodal forces, nodal force equivalents for distributed edge loadings, and nodal equivalents for body forces as

$$[k]\{\delta\} = \{f\} + \left\{f^{(p)}\right\} + \left\{f^{(b)}\right\} \tag{9.48}$$

where the stiffness matrix is given by Equation 9.24 and the load vectors are as just described. Equation 9.48 is generally applicable to finite elements used in elastic analysis. As will be learned in studying advanced finite element analysis, Equation 9.48 can be supplemented by addition of force vectors arising from plastic deformation, thermal gradients or temperature-dependent material properties, thermal swelling from radiation effects, and the dynamic effects of acceleration.

9.3 PLANE STRAIN: RECTANGULAR ELEMENT

A solid body is said to be in a state of *plane strain* if it satisfies all the assumptions of plane stress theory *except* that the body's thickness (length in the z direction) is *large* in comparison to the dimension in the xy plane. Mathematically, *plane strain* is defined as a state of loading and geometry such that

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \tag{9.49}$$

(See Appendix B for a discussion of the general stress-strain relations.)

Physically, the interpretation is that the body is so long in the z direction that the normal strain, induced by only the Poisson effect, is so small as to be negligible and, as we assume only xy-plane loadings are applied, shearing strains are also small and neglected. (One might think of plane strain as in the example of a hydroelectric dam—a large, long structure subjected to transverse loading only, not unlike a beam.) Under the prescribed conditions for plane strain, the constitutive equations for the nonzero stress components become

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu e_y]$$
$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu\varepsilon_x] \tag{9.50}$$
$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = G\gamma_{xy}$$

and, while not zero, the normal stress in the z direction is considered negligible in comparison to the other stress components.

The elastic strain energy for a body of volume V in plane strain is

$$U_e = \frac{1}{2}\iiint_V (\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy})\,\mathrm{d}V \tag{9.51}$$

which can be expressed in matrix notation as

$$U_e = \frac{1}{2}\iiint_V [\sigma_x \quad \sigma_y \quad \tau_{xy}]\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}\mathrm{d}V \tag{9.52}$$

Combining Equations 9.50 and 9.52 with considerable algebraic manipulation, the elastic strain energy is found to be

$$U_e = \frac{1}{2}\iiint_V [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}]\frac{E}{(1+\nu)(1-2\nu)}$$
$$\times\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \dfrac{1-2\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}\mathrm{d}V \tag{9.53}$$

and is similar to the case of plane stress, in that we can express the energy as

$$U_e = \frac{1}{2}\iiint_V \{\varepsilon\}^T[D]\{\varepsilon\}\,\mathrm{d}V$$

with the exception that the elastic property matrix for plane strain is defined as

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \dfrac{1-2\nu}{2}\end{bmatrix} \tag{9.54}$$

The nonzero strain components in terms of displacements are

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x}\\ \varepsilon_y &= \frac{\partial v}{\partial y}\\ \gamma_{xy} &= \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\end{aligned} \tag{9.55}$$

For a four-node rectangular element (for example only), the column matrix of strain components is expressed as

$$\{\varepsilon\} = \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x}\end{bmatrix} \begin{Bmatrix}u_1\\ u_2\\ u_3\\ u_4\\ v_1\\ v_2\\ v_3\\ v_4\end{Bmatrix} \tag{9.56}$$

in terms of the interpolation functions and the nodal displacements. As is customary, Equation 9.56 is written as

$$\{\varepsilon\} = [B]\{\delta\} \tag{9.57}$$

with $[B]$ representing the matrix of derivatives of interpolation functions and $\{\delta\}$ is the column matrix of nodal displacements. Hence, total strain energy of an element is

$$U_e = \frac{1}{2}\{\delta\}^T \iiint\limits_V [B]^T[D][B]\,\mathrm{d}V\{\delta\} = \frac{1}{2}\{\delta\}^T[k]\{\delta\} \tag{9.58}$$

and the element stiffness matrix is again given by

$$\left[k^{(e)}\right] = \iiint\limits_{V^{(e)}} [B^T][D][B]\,\mathrm{d}V^{(e)} \tag{9.59}$$

The interpolation functions for the four-node rectangular element per Equation 6.56 are

$$\begin{aligned}N_1(r,s) &= \frac{1}{4}(1-r)(1-s)\\ N_2(r,s) &= \frac{1}{4}(1+r)(1-s)\\ N_3(r,s) &= \frac{1}{4}(1+r)(1+s)\\ N_4(r,s) &= \frac{1}{4}(1-r)(1+s)\end{aligned} \tag{9.60}$$

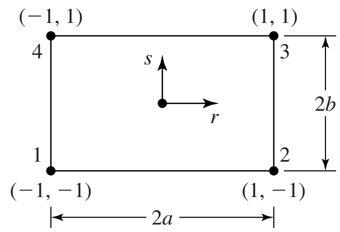


Figure 9.5 A rectangular element of width 2*a* and height 2*b*.

with the natural coordinates defined as in Figure 9.5. To compute the strain components in terms of the natural coordinates, the chain rule is applied to obtain

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial x} = \frac{1}{a}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial s}\frac{\partial s}{\partial y} = \frac{1}{b}\frac{\partial}{\partial s}\end{aligned} \tag{9.61}$$

Performing the indicated differentiations, the strain components are found to be

$$\{\varepsilon\} = \begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

$$= \begin{bmatrix} \frac{s-1}{4a} & \frac{1-s}{4a} & \frac{1+s}{4a} & -\frac{1+s}{4a} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{r-1}{4b} & -\frac{1+r}{4b} & \frac{1+r}{4b} & \frac{1-r}{4b}\\ \frac{r-1}{4b} & -\frac{1+r}{4b} & \frac{1+r}{4b} & \frac{1-r}{4b} & \frac{s-1}{4a} & \frac{1-s}{4a} & \frac{1+s}{4a} & -\frac{1+s}{4a} \end{bmatrix} \begin{Bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ v_1\\ v_2\\ v_3\\ v_4 \end{Bmatrix} \tag{9.62}$$

showing that the normal strain ε_x varies linearly in the *y* direction, normal strain ε_y varies linearly in the *x* direction, and shear strain γ_{xy} varies linearly in both coordinate directions (realizing that the natural coordinate *r* corresponds to the *x* axis and natural coordinate *s* corresponds to the *y* axis).

From Equation 9.62, the $[B]$ matrix is readily identified as

$$[B] = \begin{bmatrix} \frac{s-1}{4a} & \frac{1-s}{4a} & \frac{1+s}{4a} & -\frac{1+s}{4a} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{r-1}{4b} & -\frac{1+r}{4b} & \frac{1+r}{4b} & \frac{1-r}{4b}\\ \frac{r-1}{4b} & -\frac{1+r}{4b} & \frac{1+r}{4b} & \frac{1-r}{4b} & \frac{s-1}{4a} & \frac{1-s}{4a} & \frac{1+s}{4a} & -\frac{1+s}{4a} \end{bmatrix} \tag{9.63}$$

hence, the element stiffness matrix is given, formally, by

$$
\left[k^{(e)}\right] = \iiint\limits_{V^{(e)}} [B^T][D][B]\,\mathrm{d}V^{(e)}
$$

$$
= \frac{Etab}{(1+\nu)(1-2\nu)} \int_{-1}^{1}\int_{-1}^{1}
\begin{bmatrix}
\frac{s-1}{4a} & 0 & \frac{r-1}{4b} \\
\frac{1-s}{4a} & 0 & -\frac{1+r}{4b} \\
\frac{1+s}{4a} & 0 & \frac{1+r}{4b} \\
-\frac{1+s}{4a} & 0 & \frac{1-r}{4b} \\
0 & \frac{r-1}{4b} & \frac{s-1}{4a} \\
0 & -\frac{1+r}{4b} & \frac{1-s}{4a} \\
0 & \frac{1+r}{4b} & \frac{1+s}{4a} \\
0 & \frac{1-r}{4b} & -\frac{1+s}{4a}
\end{bmatrix}
\begin{bmatrix}
1-\nu & \nu & 0 \\
\nu & 1-\nu & 0 \\
0 & 0 & \frac{(1+\nu)(1-2\nu)}{2(1+\nu)}
\end{bmatrix}
$$

$$
\times \begin{bmatrix}
\frac{s-1}{4a} & \frac{1-s}{4a} & \frac{1+s}{4a} & -\frac{1+s}{4a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{r-1}{4b} & -\frac{1+r}{4b} & \frac{1+r}{4b} & \frac{1-r}{4b} \\
\frac{r-1}{4b} & -\frac{1+r}{4b} & \frac{1+r}{4b} & \frac{1-r}{4b} & \frac{s-1}{4a} & \frac{1-s}{4a} & \frac{1+s}{4a} & -\frac{1+s}{4a}
\end{bmatrix} \mathrm{d}r\,\mathrm{d}s \tag{9.64}
$$

The element stiffness matrix as defined by Equation 9.64 is an 8×8 symmetric matrix, which therefore, contains 36 independent terms. Hence, 36 integrations are required to obtain the complete stiffness matrix. The integrations are straightforward but algebraic tedious. Here, we develop only a single term of the stiffness matrix in detail, then discuss the more-efficient numerical methods used in finite element software packages.

If we carry out the matrix multiplications just indicated, the first diagonal term of the stiffness matrix is found (after a bit of algebra) to be

$$
k_{11}^{(e)} = \frac{Etb}{16a(1+2\nu)} \int_{-1}^{1}\int_{-1}^{1} (s-1)^2\,\mathrm{d}r\,\mathrm{d}s + \frac{Eta}{32b(1+\nu)} \int_{-1}^{1}\int_{-1}^{1} (r-1)^2\,\mathrm{d}r\,\mathrm{d}s \tag{9.65}
$$

and this term evaluates to

$$
\begin{aligned}
k_{11}^{(e)} &= \frac{Etb}{16a(1+2\nu)} \frac{2(s-1)^3}{3}\bigg|_{-1}^{1} + \frac{Eta}{32b(1+\nu)} \frac{2(r-1)^3}{3}\bigg|_{-1}^{1} \\
&= \frac{Etb}{16a(1+2\nu)}\left(\frac{16}{3}\right) + \frac{Eta}{32b(1+\nu)}\left(\frac{16}{3}\right)
\end{aligned} \tag{9.66}
$$

Note that the integrands are quadratic functions of the natural coordinates. In fact, analysis of Equation 9.64 reveals that every term of the element stiffness matrix requires integration of quadratic functions of the natural coordinates. From the earlier discussion of Gaussian integration (Chapter 6), we know that a quadratic polynomial can be integrated exactly using only two integration (or evaluation) points. As here we deal with integration in two dimensions, we must evaluate the integrand at the Gauss points

$$r_i = \pm\frac{\sqrt{3}}{3} \qquad s_j = \pm\frac{\sqrt{3}}{3}$$

with weighting factors $W_i = W_j = 1$. If we apply the numerical integration technique to evaluation of $k_{11}^{(e)}$, we obtain, as expected, the result identical to that given by Equation 9.66. More important, the Gauss integration procedure can be applied directly to Equation 9.64 to obtain the *entire* element stiffness matrix as

$$\left[k^{(e)}\right] = tab\sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j [B(r_i, s_j)]^T [D][B(r_i, s_j)] \tag{9.67}$$

where the matrix triple product is evaluated four times, in accordance with the number of integration points required. The summations and matrix multiplications required in Equation 9.67 are easily programmed and ideally suited to digital computer implementation.

While written specifically for the four-node rectangular element, Equation 9.67 is applicable to higher-order elements as well. Recall that, as the polynomial order increases, exact integration via Gaussian quadrature requires increase in both number and change in value of the integration points and weighting factors. By providing a "look-up" table of values fashioned after Table 6.1, computer implementation of Equation 9.67 can be readily adapted to higher-order elements.

We use the triangular element to illustrate plane stress and the rectangular element to illustrate plane strain. If the developments are followed clearly, it is apparent that either element can be used for either state of stress. The only difference is in the stress-strain relations exhibited by the [D] matrix. This situation is true of *any* element shape and order (in terms of number of nodes and order of polynomial interpolation functions). Our use of the examples of triangular and rectangular elements are not meant to be restrictive in any way.

9.4 ISOPARAMETRIC FORMULATION OF THE PLANE QUADRILATERAL ELEMENT

While useful for analysis of plane problems in solid mechanics, the triangular and rectangular elements just discussed exhibit shortcomings. Geometrically, the triangular element is quite useful in modeling irregular shapes having curved boundaries. However, since element strains are constant, a large number of small elements are required to obtain reasonable accuracy, particular in areas of high

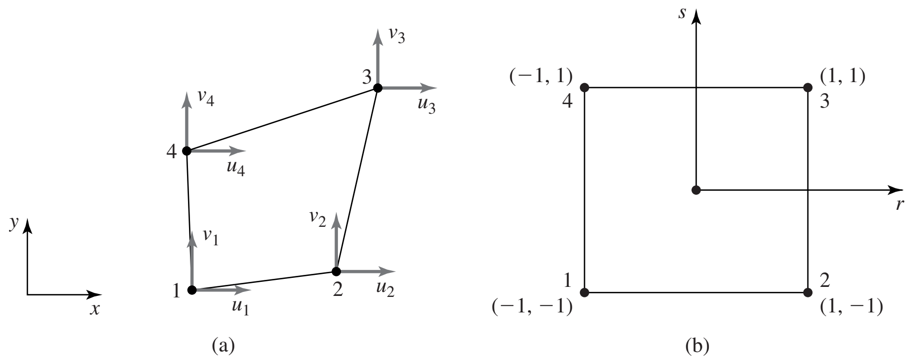


Figure 9.6
(a) A four-node, two-dimensional isoparametric element. (b) The parent element in natural coordinates.

stress gradients, such as near geometric discontinuities. In comparison, the rectangular element provides the more-reasonable linear variation of strain components but is not amenable to irregular shapes. An element having the desirable characteristic of strain variation in the element as well as the ability to closely approximate curves is the four-node quadrilateral element. We now develop the quadrilateral element using an isoparametric formulation adaptable to either plane stress or plane strain.

A general quadrilateral element is shown in Figure 9.6a, having element node numbers and nodal displacements as indicated. The coordinates of node i are (x_i, y_i) and refer to a global coordinate system. The element is formed by mapping the parent element shown in Figure 9.6b, using the procedures developed in Section 6.8. Recalling that, in the isoparametric approach, the geometric mapping functions are identical to the interpolation functions used to discretize the displacements, the geometric mapping is defined by

$$
\begin{aligned}
x &= \sum_{i=1}^{4} N_i(r, s) x_i \\
y &= \sum_{i=1}^{4} N_i(r, s) y_i
\end{aligned}
\tag{9.68}
$$

and the interpolation functions are as given in Equation 9.60, so that the displacements are described as

$$
\begin{aligned}
u(x, y) &= \sum_{i=1}^{4} N_i(r, s) u_i \\
v(x, y) &= \sum_{i=1}^{4} N_i(r, s) v_i
\end{aligned}
\tag{9.69}
$$

Now, the mathematical complications arise in computing the strain components as given by Equation 9.55 and rewritten here as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{9.70}$$

Using Equation 6.83 with $\phi = u$, we have

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \end{aligned} \tag{9.71}$$

with similar expressions for the partial derivative of the v displacement. Writing Equation 9.71 in matrix form

$$\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial s} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} \tag{9.72}$$

and the Jacobian matrix is identified as

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \tag{9.73}$$

as in Equation 6.83. Note that, per the geometric mapping of Equation 9.68, the components of $[J]$ are known as functions of the partial derivatives of the interpolation functions and the nodal coordinates in the xy plane. For example,

$$J_{11} = \frac{\partial x}{\partial r} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial r} x_i = \frac{1}{4}[(s-1)x_1 + (1-s)x_2 + (1+s)x_3 - (1+s)x_4] \tag{9.74}$$

a first-order polynomial in the natural (mapping) coordinate s. The other terms are similarly first-order polynomials.

Formally, Equation 9.72 can be solved for the partial derivatives of displacement component u with respect to x and y by multiplying by the inverse of the Jacobian matrix. As noted in Chapter 6, finding the inverse of the Jacobian matrix in algebraic form is not an enviable task. Instead, numerical methods are used, again based on Gaussian quadrature, and the remainder of the derivation here is toward that end. Rather than invert the Jacobian matrix, Equation 9.72

can be solved via Cramer's rule. Application of Cramer's rule results in

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\begin{vmatrix} \dfrac{\partial u}{\partial r} & J_{12} \\ \dfrac{\partial u}{\partial s} & J_{22} \end{vmatrix}}{|J|} = \frac{1}{|J|}[J_{22} \quad -J_{12}]\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial s} \end{Bmatrix} \\
\frac{\partial u}{\partial y} &= \frac{\begin{vmatrix} J_{11} & \dfrac{\partial u}{\partial r} \\ J_{21} & \dfrac{\partial u}{\partial s} \end{vmatrix}}{|J|} = \frac{1}{|J|}[-J_{21} \quad +J_{11}]\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial s} \end{Bmatrix}
\end{aligned}
\tag{9.75}
$$

or, in a more compact form,

$$
\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial s} \end{Bmatrix}
\tag{9.76}
$$

The determinant of the Jacobian matrix $|J|$ is commonly called simply the *Jacobian.*

Since the interpolation functions are the same for both displacement components, an identical procedure results in

$$
\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}\begin{Bmatrix} \dfrac{\partial v}{\partial r} \\ \dfrac{\partial v}{\partial s} \end{Bmatrix}
\tag{9.77}
$$

for the partial derivatives of the v displacement component with respect to global coordinates.

Let us return to the problem of computing the strain components per Equation 9.70. Utilizing Equations 9.76 and 9.77, the strain components are expressed as

$$
\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial s} \\ \dfrac{\partial v}{\partial r} \\ \dfrac{\partial v}{\partial s} \end{Bmatrix} = [G]\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial s} \\ \dfrac{\partial v}{\partial r} \\ \dfrac{\partial v}{\partial s} \end{Bmatrix}
\tag{9.78}
$$

with what we will call the *geometric mapping matrix,* defined as

$$[G] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \tag{9.79}$$

We must expand the column matrix on the extreme right-hand side of Equation 9.78 in terms of the discretized approximation to the displacements. Via Equation 9.69, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial s} \\ \dfrac{\partial v}{\partial r} \\ \dfrac{\partial v}{\partial s} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial r} & \dfrac{\partial N_2}{\partial r} & \dfrac{\partial N_3}{\partial r} & \dfrac{\partial N_4}{\partial r} & 0 & 0 & 0 & 0 \\ \dfrac{\partial N_1}{\partial s} & \dfrac{\partial N_2}{\partial s} & \dfrac{\partial N_3}{\partial s} & \dfrac{\partial N_4}{\partial s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{\partial N_1}{\partial r} & \dfrac{\partial N_2}{\partial r} & \dfrac{\partial N_3}{\partial r} & \dfrac{\partial N_4}{\partial r} \\ 0 & 0 & 0 & 0 & \dfrac{\partial N_1}{\partial s} & \dfrac{\partial N_2}{\partial s} & \dfrac{\partial N_3}{\partial s} & \dfrac{\partial N_4}{\partial s} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} \tag{9.80}$$

where we reemphasize that the indicated partial derivatives are known functions of the natural coordinates of the parent element. For shorthand notation, Equation 9.80 is rewritten as

$$\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial s} \\ \dfrac{\partial v}{\partial r} \\ \dfrac{\partial v}{\partial s} \end{Bmatrix} = [P]\{\delta\} \tag{9.81}$$

in which $[P]$ is the matrix of partial derivatives and $\{\delta\}$ is the column matrix of nodal displacement components.

Combining Equations 9.78 and 9.81, we obtain the sought-after relation for the strain components in terms of nodal displacement components as

$$\{\varepsilon\} = [G][P]\{\delta\} \tag{9.82}$$

and, by analogy with previous developments, matrix $[B] = [G][P]$ has been determined such that

$$\{\varepsilon\} = [B]\{\delta\} \tag{9.83}$$

and the element stiffness matrix is defined by

$$\left[k^{(e)}\right] = t \int_A [B]^T [D][B]\,\mathrm{d}A \tag{9.84}$$

with t representing the constant element thickness, and the integration is performed over the area of the element (in the physical xy plane). In Equation 9.84, the stiffness may represent a plane stress element or a plane strain element, depending on whether the material property matrix $[D]$ is defined by Equation 9.6 or 9.54, respectively. (Also note that, for plane strain, it is customary to take the element thickness as unity.)

The integration indicated by Equation 9.84 are in the x-y global space, but the $[B]$ matrix is defined in terms of the natural coordinates in the parent element space. Therefore, a bit more analysis is required to obtain a final form. In the physical space, we have $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$, but we wish to integrate using the natural coordinates over their respective ranges of -1 to $+1$. In the case of the four-node rectangular element, the conversion is straightforward, as x is related only to r and y is related only to s, as indicated in Equation 9.61. In the isoparametric case at hand, the situation is not quite so simple. The derivation is not repeated here, but it is shown in many calculus texts [1] that

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}r\,\mathrm{d}s \tag{9.85}$$

hence, Equation 9.84 becomes

$$\left[k^{(e)}\right] = t\int_A [B]^T[D][B]\,|J|\,\mathrm{d}r\,\mathrm{d}s = t\int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]\,|J|\,\mathrm{d}r\,\mathrm{d}s \tag{9.86}$$

As noted, the terms of the $[B]$ matrix are known functions of the natural coordinates, as is the Jacobian $|J|$. The terms in the stiffness matrix represented by Equation 9.86, in fact, are integrals of ratios of polynomials and the integrations are very difficult, usually impossible, to perform exactly. Instead, Gaussian quadrature is used and the integrations are replaced with sums of the integrand evaluated at specified Gauss points as defined in Chapter 6. For p integration points in the variable r and q integration points in the variable s, the stiffness matrix is approximated by

$$\left[k^{(e)}\right] = t\sum_{i=1}^{p}\sum_{j=1}^{q} W_i W_J[B(r_i, s_j)]^T[D][B(r_i, s_j)]|J(r_i, s_j)|\mathrm{d}r\,\mathrm{d}s \tag{9.87}$$

Since $[B]$ includes the determinant of the Jacobian matrix in the denominator, the numerical integration does not necessarily result in an exact solution, since the ratio of polynomials is not necessarily a polynomial. Nevertheless, the Gaussian procedure is used for this element, as if the integrand is a quadratic in both r and s, with good results. In such case, we use two Gauss points for each variable, as is illustrated in the following example.

EXAMPLE 9.3

Evaluate the stiffness matrix for the isoparametric quadrilateral element shown in Figure 9.7 for plane stress with $E = 30(10)^6$ psi, $\nu = 0.3$, $t = 1$ in. Note that the properties are those of steel.

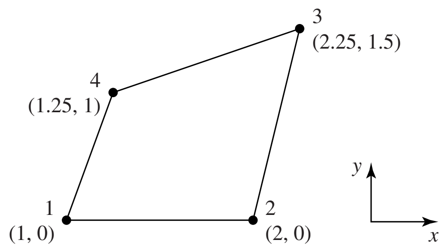


Figure 9.7 Dimensions are in inches. Axes are shown for orientation only.

■ Solution

The mapping functions are

$$x(r,s) = \frac{1}{4}[(1-r)(1-s)(1) + (1+r)(1-s)(2) + (1+r)(1+s)(2.25) + (1-r)(1+s)(1.25)]$$

$$y(r,s) = \frac{1}{4}[(1-r)(1-s)(0) + (1+r)(1-s)(0) + (1+r)(1+s)(1.5) + (1-r)(1+s)(1)]$$

and the terms of the Jacobian matrix are

$$J_{11} = \frac{\partial x}{\partial r} = \frac{1}{2}$$

$$J_{12} = \frac{\partial y}{\partial r} = \frac{1}{4}(0.5 - 0.5s)$$

$$J_{21} = \frac{\partial x}{\partial s} = \frac{1}{2}$$

$$J_{22} = \frac{\partial y}{\partial s} = \frac{1}{4}(2.5 - 0.5r)$$

and the determinant is

$$|J| = J_{11}J_{22} - J_{12}J_{21} = \frac{1}{16}(4 - r + s)$$

Therefore, the geometric matrix $[G]$ of Equation 9.79 is known in terms of ratios of monomials in r and s as

$$[G] = \frac{4}{4-r+s}\begin{bmatrix} 2.5-0.5r & -(0.5-0.5s) & 0 & 0 \\ 0 & 0 & -2 & 2 \\ -2 & 2 & 2.5-0.5r & -(0.5-0.5s) \end{bmatrix}$$

For plane stress with the values given, the material property matrix is

$$[D] = 32.97(10)^6 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \text{psi}$$

Next, we note that, since the matrix of partial derivatives $[P]$ as defined in Equation 9.81 is also composed of monomials in r and s,

$$[P] = \frac{1}{4}\begin{bmatrix} s-1 & 1-s & 1+s & -(1+s) & 0 & 0 & 0 & 0 \\ r-1 & -(1+r) & 1+r & 1-r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s-1 & 1-s & 1+s & -(1+s) \\ 0 & 0 & 0 & 0 & r-1 & -(1+r) & 1+r & 1-r \end{bmatrix}$$

the stiffness matrix of Equation 9.86 is no more than quadratic in the natural coordinates. Hence, we select four integration points given by

$$r_i = s_j = \pm\frac{\sqrt{3}}{3}$$

and weighting factors

$$W_i = W_j = 1.0$$

per Table 6.1.

The element stiffness matrix is then given by

$$\left[k^{(e)}\right] = t\sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j [B(r_i, s_j)]^T [D][B(r_i, s_j)]|J(r_i, s_j)|$$

The numerical results for this example are obtained via a computer program written in MATLAB using the built-in matrix functions of that software package. The stiffness matrix is calculated to be

$$\left[k^{(e)}\right] = \begin{bmatrix} 2305 & -1759 & -617 & 72 & 798 & -152 & -214 & -432 \\ -1759 & 1957 & 471 & -669 & -52 & -522 & 14 & 560 \\ -617 & 471 & 166 & -19 & -214 & 41 & 57 & 116 \\ 72 & -669 & -19 & 616 & -533 & 633 & 143 & -244 \\ 798 & -52 & -214 & -533 & 1453 & -169 & -389 & -895 \\ -152 & -522 & -41 & 633 & -169 & 993 & 45 & -869 \\ -214 & 14 & 57 & 143 & -389 & 45 & 104 & 240 \\ -432 & 560 & 116 & -244 & -895 & -869 & 240 & 1524 \end{bmatrix} 10^3 \text{ lb/in.}$$

EXAMPLE 9.4

A classic example of plane stress analysis is shown in Figure 9.8a. A uniform thin plate with a central hole of radius a is subjected to uniaxial stress σ_0. Use the finite element method to determine the stress concentration factor given the physical data $\sigma_0 = 1000$ psi, $a = 0.5$ in., $h = 3$ in., $w = 6$ in., $E = 10(10)^6$ psi, and Poisson's ratio = 0.3.

■ Solution

The solution for this example is obtained using commercial finite element software with plane quadrilateral elements. The initial (coarse) element mesh, shown in Figure 9.8b, is composed of 33 elements. Note that the symmetry conditions have been used to reduce the model to quarter-size and the corresponding boundary conditions are as shown on the figure. For this model, the maximum stress (as expected) is calculated to occur at node 1 (at the top of the hole) and has a magnitude of 3101 psi.

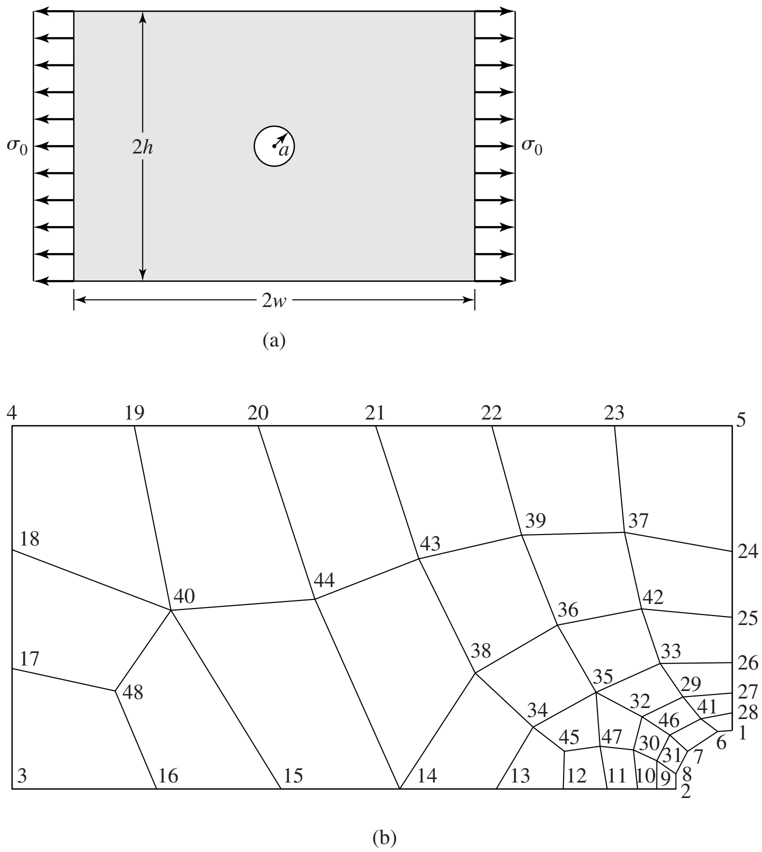
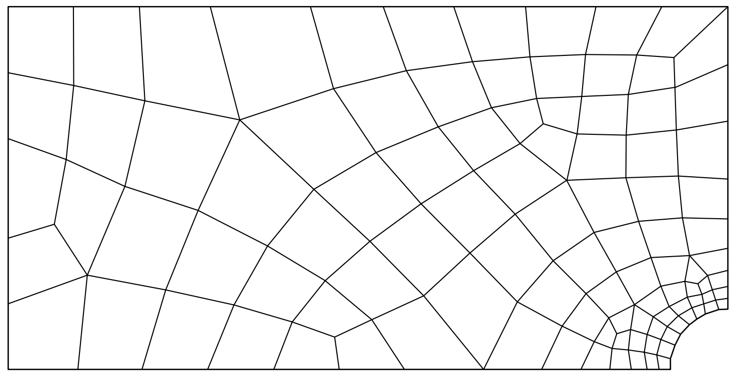


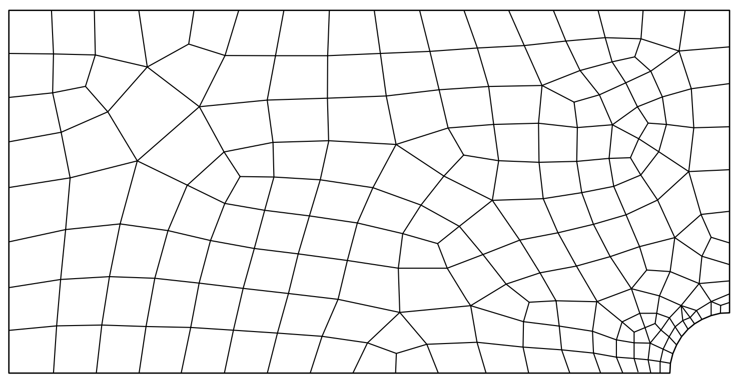
Figure 9.8
(a) A uniformly loaded plate in plane stress with a central hole of radius *a*. (b) A coarse finite element mesh using quadrilateral elements. Node numbers are as shown (31 elements).

To examine the solution convergence, a refined model is shown in Figure 9.8c, using 101 elements. For this model, the maximum stress also occurs at node 1 and has a calculated magnitude of 3032 psi. Hence, between the two models, the maximum stress values changed on the order of 2.3 percent. It is interesting to note that the maximum displacement given by the two models is essentially the same. This observation reinforces the need to examine the derived variables for convergence, not simply the directly computed variables.

As a final step in examining the convergence, the model shown in Figure 9.8d containing 192 elements is also solved. (The node numbers are eliminated for clarity.) The maximum computed stress, again at node 1, is 3024 psi, a miniscule change relative to the previous model, so we conclude that convergence has been attained. (The change in maximum displacement is essentially nil.) Hence, we conclude that the stress concentration factor $K_t = \sigma_{max}/\sigma_0 = 3024/1000 = 3.024$ is applicable to the geometry and loading of this example. It is interesting to note that the theoretical (hence, the subscript *t*)



(c)



(d)

Figure 9.8 (*Continued*)
(c) Refined mesh of 101 elements. Node numbers are removed for clarity. (d) An additional refined mesh with 192 elements.

stress concentration factor for this problem as computed by the mathematical theory of elasticity is exactly 3. The same result is shown in many texts on machine design and stress analysis [2].

9.5 AXISYMMETRIC STRESS ANALYSIS

The concept of axisymmetry is discussed in Chapter 6 in terms of general interpolation functions. Here, we specialize the axisymmetric concept to problems of elastic stress analysis. To satisfy the conditions for axisymmetric stress, the problem must be such that

1. The solid body under stress must be a solid of revolution; by convention, the axis of revolution is the z axis in a cylindrical coordinate system (r, θ, z).
2. The loading of the body is symmetric about the z axis.

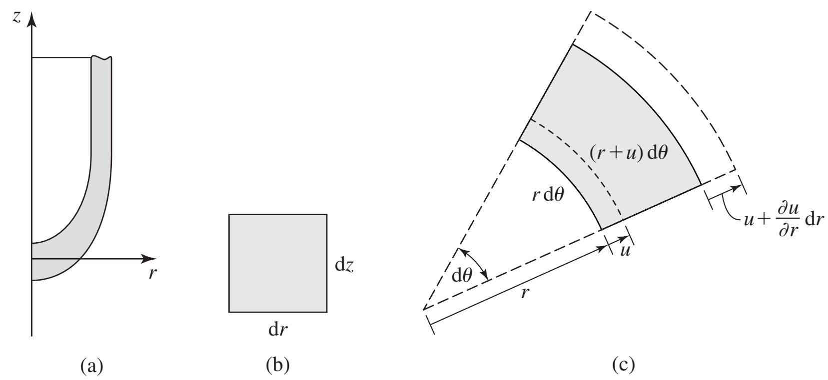


Figure 9.9
(a) Cross section of an axisymmetric body. (b) Differential element in an *rz* plane. (c) Differential element in an r-θ plane illustrating tangential deformation. Dashed lines represent deformed positions.

3. All boundary (constraint) conditions are symmetric about the z axis.
4. Materials properties are also symmetric (automatically satisfied by a linearly elastic, homogeneous, isotropic material).

If these conditions are satisfied, the displacement field is independent of the tangential coordinate θ, and hence the stress analysis is mathematically two-dimensional, even though the physical problem is three-dimensional. To develop the axisymmetric equations, we examine Figure 9.9a, representing a solid of revolution that satisfies the preceding requirements. Figure 9.9b is a differential element of the body in the rz plane; that is, any section through the body for which θ is constant. We cannot ignore the tangential coordinate completely, however, since as depicted in Figure 9.9c, there is strain in the tangential direction (recall the basic definition of hoop stress in thin-walled pressure vessels from mechanics of materials). Note that, in the radial direction, the element undergoes displacement, which introduces increase in circumference and associated circumferential strain.

We denote the radial displacement as u, the tangential (circumferential) displacement as v, and the axial displacement as w. From Figure 9.9c, the radial strain is

$$\varepsilon_r = \frac{1}{dr}\left(u + \frac{\partial u}{\partial r}\,dr - u\right) = \frac{\partial u}{\partial r} \tag{9.88}$$

The axial strain is

$$\varepsilon_z = \frac{1}{dz}\left(w + \frac{\partial w}{\partial z}\,dz - w\right) = \frac{\partial w}{\partial z} \tag{9.89}$$

and these relations are as expected, since the rz plane is effectively the same as a rectangular coordinate system. In the circumferential direction, the differential element undergoes an expansion defined by considering the original arc length versus the deformed arc length. Prior to deformation, the arc length is $\mathrm{d}s = r\,\mathrm{d}\theta$, while after deformation, arc length is $\mathrm{d}s = (r+u)\,\mathrm{d}\theta$. The tangential strain is

$$\varepsilon_\theta = \frac{(r+u)(\mathrm{d}\theta) - r\,\mathrm{d}\theta}{r\,\mathrm{d}\theta} = \frac{u}{r} \tag{9.90}$$

and we observe that, even though the problem is independent of the tangential coordinate, the tangential strain must be considered in the problem formulation. Note that, if $r = 0$, the preceding expression for the tangential strain is troublesome mathematically, since division by zero is indicated. The situation occurs, for example, if we examine stresses in a rotating solid body, in which case the stresses are induced by centrifugal force (normal acceleration). Additional discussion of this problem is included later when we discuss element formulation.

Additionally, the shear strain components are

$$\begin{aligned} \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ \gamma_{r\theta} &= 0 \\ \gamma_{\theta z} &= 0 \end{aligned} \tag{9.91}$$

If we substitute the strain components into the generalized stress-strain relations of Appendix B (and, in this case, we utilize $\theta = y$), we obtain

$$\begin{aligned} \sigma_r &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_r + \nu(\varepsilon_\theta + \varepsilon_z)] \\ \sigma_\theta &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_\theta + \nu(\varepsilon_r + \varepsilon_z)] \\ \sigma_z &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_z + \nu(\varepsilon_r + \varepsilon_\theta)] \\ \tau_{rz} &= \frac{E}{2(1+\nu)}\gamma_{rz} = G\gamma_{rz} \end{aligned} \tag{9.92}$$

For convenience in finite element development, Equation 9.92 is expressed in matrix form as

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{ez} \end{Bmatrix} \tag{9.93}$$

in which we identify the material property matrix for axisymmetric elasticity as

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \tag{9.94}$$

9.5.1 Finite Element Formulation

Recall from the general discussion of interpolation functions in Chapter 6 that essentially any two-dimensional element can be used to generate an axisymmetric element. As there is, by definition, no dependence on the θ coordinate and no circumferential displacement, the displacement field for the axisymmetric stress problem can be expressed as

$$
\begin{aligned}
u(r, z) &= \sum_{i=1}^{M} N_i(r, z) u_i \\
w(r, z) &= \sum_{i=1}^{M} N_i(r, z) w_i
\end{aligned}
\tag{9.95}
$$

with u_i and w_i representing the nodal radial and axial displacements, respectively. For illustrative purposes, we now assume the case of a three-node triangular element.

The strain components become

$$
\begin{aligned}
\varepsilon_r &= \frac{\partial u}{\partial r} = \sum_{i=1}^{3} \frac{\partial N_i}{\partial r} u_i \\
\varepsilon_\theta &= \frac{u}{r} = \sum_{i=1}^{3} \frac{N_i}{r} u_i \\
\varepsilon_z &= \frac{\partial w}{\partial z} = \sum_{i=1}^{3} \frac{\partial N_i}{\partial z} w_i \\
\gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = \sum_{i=1}^{3} \frac{\partial N_i}{\partial z} u_i + \sum_{i=1}^{3} \frac{\partial N_i}{\partial r} w_i
\end{aligned}
\tag{9.96}
$$

and these are conveniently expressed in the matrix form

$$
\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix} =
\begin{bmatrix}
\dfrac{\partial N_1}{\partial r} & \dfrac{\partial N_2}{\partial r} & \dfrac{\partial N_3}{\partial r} & 0 & 0 & 0 \\
\dfrac{N_1}{r} & \dfrac{N_2}{r} & \dfrac{N_3}{r} & 0 & 0 & 0 \\
0 & 0 & 0 & \dfrac{\partial N_1}{\partial z} & \dfrac{\partial N_2}{\partial z} & \dfrac{\partial N_3}{\partial z} \\
\dfrac{\partial N_1}{\partial z} & \dfrac{\partial N_2}{\partial z} & \dfrac{\partial N_3}{\partial z} & \dfrac{\partial N_1}{\partial r} & \dfrac{\partial N_2}{\partial r} & \dfrac{\partial N_3}{\partial r}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ w_1 \\ w_2 \\ w_3 \end{Bmatrix}
\tag{9.97}
$$

In keeping with previous developments, Equation 9.97 is denoted $\{\varepsilon\} = [B]\{\delta\}$ with $[B]$ representing the 4×6 matrix involving the interpolation functions. Thus total strain energy of the elements, as described by Equation 9.15 or 9.58

and the stiffness matrix, is

$$\left[k^{(e)}\right] = \iiint\limits_{V^{(e)}} [B]^T [D][B]\, \mathrm{d}V^{(e)} \tag{9.98}$$

While Equation 9.98 is becoming rather familiar, a word or two of caution is appropriate. Recall in particular that, although the interpolation functions used here are two dimensional, the axisymmetric element is truly three dimensional (toroidal). Second, the element is not a constant strain element, owing to the inverse variation of ε_θ with radial position, so the integrand in Equation 9.98 is not constant. Finally, note that [D] is significantly different in comparison to the counterpart material property matrices for plane stress and plane strain. Taking the first observation into account and recalling Equation 6.93, the stiffness matrix is defined by

$$\left[k^{(e)}\right] = 2\pi \iint\limits_{A^{(e)}} [B]^T [D][B] r\, \mathrm{d}r\, \mathrm{d}z \tag{9.99}$$

and is a 6×6 symmetric matrix requiring, in theory, evaluation of 21 integrals. Explicit term-by-term integration is not recommended, owing to the algebraic complexity. When high accuracy is required, Gauss-type numerical integration using integration points specifically determined for triangular regions [3] is used. Another approach is to evaluate matrix [B] at the centroid of the element in an rz plane. In this case, the matrices in the integrand become constant and the stiffness matrix is approximated by

$$\left[k^{(e)}\right] \approx 2\pi \bar{r} A [\bar{B}]^T [D][\bar{B}] \tag{9.100}$$

Of course, the accuracy of the approximation improves as element size is decreased.

Referring to a previous observation, formulation of the [B] matrix is troublesome if $r = 0$ is included in the domain. In this occurrence, three terms of Equation 9.97 "blow up," owing to division by zero. If the stiffness matrix is evaluated using the centroidal approximation of Equation 9.100, the problem is avoided, since the radial coordinate of the centroid of any element cannot be zero in an axisymmetric finite element model. Nevertheless, radial and tangential strain and stress components cannot be evaluated at nodes for which $r = 0$. Physically, we know that the radial and tangential displacements at $r = 0$ in an axisymmetric problem must be zero. Mathematically, the observation is not accounted for in the general finite element formulation, which is for an arbitrary domain. One technique for avoiding the problem is to include a hole, coinciding with the z axis and having a small, but finite radius [4].

9.5.2 Element Loads

Axisymmetric problems often involve surface forces in the form of internal or external pressure and body forces arising from rotation of the body (centrifugal

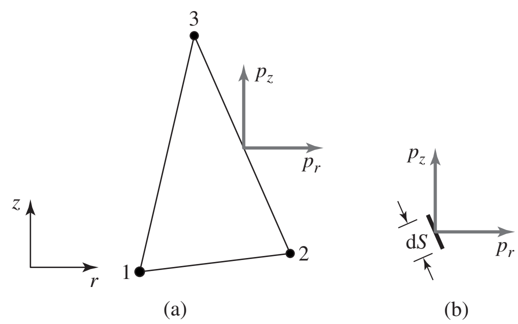


Figure 9.10
(a) Axisymmetric element. (b) Differential length of the element edge.

force) and gravity. In each case, the external influences are reduced to nodal forces using the work equivalence concept previously introduced.

The triangular axisymmetric element shown in Figure 9.10a is subjected to pressures p_r and p_z in the radial and axial directions, respectively. The equivalent nodal forces are determined by analogy with Equation 9.39, with the notable exception depicted in Figure 9.10b, showing a differential length dS of the element edge in question. As dS is located a radial distance r from the axis of symmetry, the area on which the pressure components act is $2\pi r\,\mathrm{d}S$. The nodal forces are given by

$$\left\{f^{(p)}\right\} = \begin{Bmatrix} f_r^{(p)} \\ f_z^{(p)} \end{Bmatrix} = 2\pi \int_S [N]^T \begin{Bmatrix} p_r \\ p_z \end{Bmatrix} r\,\mathrm{d}S \tag{9.101}$$

and the path of integration S is the element edge. In this expression, $[N]^T$ is as defined by Equation 9.40.

EXAMPLE 9.5

Calculate the nodal forces corresponding to a uniform radial pressure $p_r = 10$ psi acting as shown on the axisymmetric element in Figure 9.11.

■ Solution

As we have pressure on one face only and no axial pressure, we immediately observe that

$$f_{r2} = f_{z1} = f_{z2} = f_{z3} = 0$$

The nonzero terms are

$$f_{r1} = 2\pi \int_S N_1 p_r r\,\mathrm{d}S$$

$$f_{r3} = 2\pi \int_S N_3 p_r r\,\mathrm{d}S$$

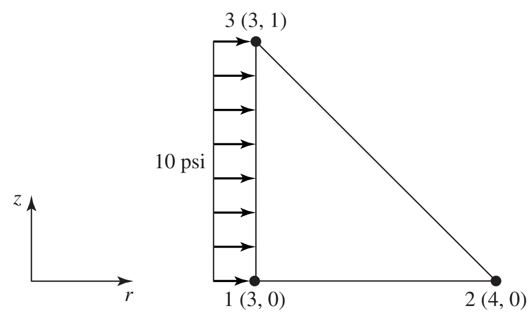


Figure 9.11 Uniform radial pressure. Dimensions are in inches.

Using Equation 9.28 with r, z in place of x, y, the interpolation functions are

$$N_1 = 4 - r - z$$
$$N_2 = r - 3$$
$$N_3 = z$$

and along the integration path ($r = 3$), we have

$$N_1 = 1 - z$$
$$N_2 = 0$$
$$N_3 = z$$

If the integration path is *from* node 1 *to* node 3, then $\mathrm{d}S = \mathrm{d}z$ and

$$f_{r1} = 2\pi(10)(3)\int_0^1 z\,\mathrm{d}z = 30\pi \text{ lb}$$

$$f_{r3} = 2\pi(10)(3)\int_0^1 (1 - z)\,\mathrm{d}z = 30\pi \text{ lb}$$

Note that, if the integration path is taken in the opposite sense (i.e., *from* node 3 *to* node 2), then $\mathrm{d}S = -\mathrm{d}z$ and the same results are obtained.

Body forces acting on axisymmetric elements are accounted for in a manner similar to that discussed for the plane stress element, while taking into consideration the geometric differences. If body forces (force per unit mass) R_B and Z_B act in the radial and axial directions, respectively, the equivalent nodal forces are calculated as

$$\left\{f^{(B)}\right\} = 2\pi\rho \int_{A^{(e)}} [N]^T \begin{Bmatrix} R_B \\ Z_B \end{Bmatrix} r\,\mathrm{d}r\,\mathrm{d}z \tag{9.102}$$

For the three-node triangular element, $[N]^T$ would again be as given in Equation 9.40. Extension to other element types is similar.

Generally, radial body force arises from rotation of an axisymmetric body about the z axis. For constant angular velocity ω, the radial body force component R_B is equal to the magnitude of the normal acceleration component $r\omega^2$ and directed in the positive radial direction.

EXAMPLE 9.6

The axisymmetric element of Figure 9.11 is part of a body rotating with angular velocity 10 rad/s about the z axis and subjected to gravity in the negative z direction. Compute the equivalent nodal forces. Density is $7.3(10)^{-4}$ lb-s^2/in.^{4}

■ Solution

For the stated conditions, we have

$$R_B = r\omega^2 = 100r \text{ in./s}^2$$
$$Z_B = -g = -386.4 \text{ in./s}^2$$

Using the interpolation functions as given in Example 9.5,

$$f_{r1} = 2\pi\rho \int_A N_1 R_B r \,\mathrm{d}r\,\mathrm{d}z = 2\pi\rho(100) \int_3^4 \int_0^{4-r} (4-r-z)r^2 \,\mathrm{d}z\,\mathrm{d}r = 0.84 \text{ lb}$$

$$f_{r2} = 2\pi\rho \int_A N_2 R_B r \,\mathrm{d}r\,\mathrm{d}z = 2\pi\rho(100) \int_3^4 \int_0^{4-r} (r-3)r^2 \,\mathrm{d}z\,\mathrm{d}r = 0.98 \text{ lb}$$

$$f_{r3} = 2\pi\rho \int_A N_3 R_B r \,\mathrm{d}r\,\mathrm{d}z = 2\pi\rho(100) \int_3^4 \int_0^{4-r} zr^2 \,\mathrm{d}z\,\mathrm{d}r = 0.84 \text{ lb}$$

$$f_{z1} = 2\pi\rho \int_A N_1 Z_B r \,\mathrm{d}r\,\mathrm{d}z = -2\pi\rho(386.4) \int_3^4 \int_0^{4-r} (4-r-z)r \,\mathrm{d}z\,\mathrm{d}r = -1.00 \text{ lb}$$

$$f_{z2} = 2\pi\rho \int_A N_2 Z_B r \,\mathrm{d}r\,\mathrm{d}z = -2\pi\rho(386.4) \int_3^4 \int_0^{4-r} (r-3)r \,\mathrm{d}z\,\mathrm{d}r = -1.08 \text{ lb}$$

$$f_{z3} = 2\pi\rho \int_A N_3 Z_B r \,\mathrm{d}r\,\mathrm{d}z = -2\pi\rho(386.4) \int_3^4 \int_0^{4-r} zr \,\mathrm{d}z\,\mathrm{d}r = -1.00 \text{ lb}$$

The integrations required to obtain the given results are straightforward but algebraically tedious. Another approach that can be used and is increasingly accurate for decreasing element size is to evaluate the body forces and the integrand at the centroid of the cross section of the element area as an approximation. Using this approximation, it can be shown that

$$\int_A N_i(\bar{r}, \bar{z})\bar{r} \,\mathrm{d}z\,\mathrm{d}r = \frac{\bar{r}A}{3} \qquad i = 1, 3$$

so the body forces are allocated equally to each node. For the present example, the result is

$$f_{r1} = f_{r2} = f_{r3} = 0.88 \text{ lb}$$
$$f_{z1} = f_{z2} = f_{z3} = -1.03 \text{ lb}$$

Note that, within the numerical accuracy used here, the total radial force and the total axial force are the same for the two methods.

9.6 GENERAL THREE-DIMENSIONAL STRESS ELEMENTS

While the conditions of plane stress, plane strain, and axisymmetry are frequently encountered, more often than not the geometry of a structure and the applied loads are such that a general three-dimensional state of stress exists. In the general case, there are three displacement components u, v, and w in the directions of the x, y, and z axes, respectively, and six strain components given by (Appendix B)

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \end{Bmatrix} \tag{9.103}$$

For convenience of presentation, the strain-displacement relations of Equation 9.103 can be expressed as

$$\{\varepsilon\} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [L] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{9.104}$$

and matrix $[L]$ is the 6×3 matrix of derivative operators.

The stress-strain relations, Equation B.12, are expressed in matrix form as

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}$$

$$= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \{\varepsilon\} = [D]\{\varepsilon\} \tag{9.105}$$

Note that, for the general case, the material property matrix $[D]$ is a 6×6 matrix involving only the elastic modulus and Poisson's ratio (we continue to restrict the presentation to linear elasticity). Also note that the displacement components are continuous functions of the Cartesian coordinates.

9.6.1 Finite Element Formulation

Following the general procedure established in the context of two-dimensional elements, a three-dimensional elastic stress element having M nodes is formulated by first discretizing the displacement components as

$$\begin{aligned} u(x, y, z) &= \sum_{i=1}^{M} N_i(x, y, z) u_i \\ v(x, y, z) &= \sum_{i=1}^{M} N_i(x, y, z) v_i \\ w(x, y, z) &= \sum_{i=1}^{M} N_i(x, y, z) w_i \end{aligned} \tag{9.106}$$

As usual, the Cartesian nodal displacements are u_i, v_i, and w_i and $N_i(x, y, z)$ is the interpolation function associated with node i. At this point, we make no assumption regarding the element shape or number of nodes. Instead, we simply note that the interpolation functions may be any of those discussed in Chapter 6 for three-dimensional elements.

Introducing the vector (column matrix) of nodal displacements,

$$\{\delta\} = [\, u_1 \quad u_2 \quad \cdots \quad u_M \quad v_1 \quad v_2 \quad \cdots \quad v_M \quad w_1 \quad w_2 \quad \cdots \quad w_M \,]^T \tag{9.107}$$

the discretized representation of the displacement field can be written in matrix form as

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} [N] & [0] & [0] \\ [0] & [N] & [0] \\ [0] & [0] & [N] \end{bmatrix} \{\delta\} = [N_3]\{\delta\} \tag{9.108}$$

In the last equation, each submatrix $[N]$ is the $1 \times M$ row matrix of interpolation functions

$$[N] = [N_1 \quad N_2 \quad \cdots \quad N_M] \tag{9.109}$$

so the matrix we have chosen to denote as $[N_3]$ is a $3 \times 3M$ matrix composed of the interpolation functions and many zero values. (Before proceeding, we emphasize that the order of nodal displacements in Equation 9.107 is convenient for purposes of development but *not* efficient for computational purposes. Much higher computational efficiency is obtained in the model solution phase if the displacement vector is defined as $\{\delta\} = [u_1\ v_1\ w_1\ u_2\ v_2\ w_2\ \cdots\ u_M\ v_M\ w_M]^T$.)

Recalling Equations 9.10 and 9.19, total potential energy of an element can be expressed as

$$\Pi = U_e - W = \frac{1}{2} \iiint\limits_V \{\varepsilon\}^T [D] \{\varepsilon\}\, \mathrm{d}V - \{\delta\}^T \{f\} \tag{9.110}$$

The element nodal force vector is defined in the column matrix

$$\{f\} = [f_{1x} \quad f_{2x} \quad \cdots \quad f_{Mx} \quad f_{1y} \quad f_{2y} \quad \cdots \quad f_{My} \quad f_{1z} \quad f_{2z} \quad \cdots \quad f_{Mz}]^T \tag{9.111}$$

and may include the effects of concentrated forces applied at the nodes, nodal equivalents to body forces, and nodal equivalents to applied pressure loadings.

Considering the foregoing developments, Equation 9.110 can be expressed (using Equations 9.104, 9.105, and 9.108), as

$$\Pi = U_e - W = \frac{1}{2} \iiint\limits_V \delta^T [L]^T [N_3]^T [D][L][N_3]\{\delta\}\, \mathrm{d}V - \{\delta\}^T \{f\} \tag{9.112}$$

As the nodal displacement components are independent of the integration over the volume, Equation 9.112 can be written as

$$\Pi = U_e - W = \frac{1}{2}\{\delta\}^T \iiint\limits_V [L]^T [N_3]^T [D][L][N_3]\, \mathrm{d}V \{\delta\} - \{\delta\}^T \{f\} \tag{9.113}$$

which is in the form

$$\Pi = U_e - W = \frac{1}{2}\{\delta\}^T \iiint\limits_V [B]^T [D][B]\, \mathrm{d}V \{\delta\} - \{\delta\}^T \{f\} \tag{9.114}$$

In Equation 9.114, the strain-displacement matrix is given by

$$[B] = [L][N_3] = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} [N] & [0] & [0] \\ [0] & [N] & [0] \\ [0] & [0] & [N] \end{bmatrix} \tag{9.115}$$

and is observed to be a $6 \times 3M$ matrix composed of the first partial derivatives of the interpolation functions.

Application of the principle of minimum potential energy to Equation 9.114 yields, in analogy with Equation 9.22,

$$\iiint\limits_V [B]^T[D][B]\,\mathrm{d}V\{\delta\} = \{f\} \tag{9.116}$$

as the system of nodal equilibrium equation for a general three-dimensional stress element. From Equation 9.116, we identify the element stiffness matrix as

$$[k] = \iiint\limits_V [B]^T[D][B]\,\mathrm{d}V \tag{9.117}$$

and the element stiffness matrix so defined is a $3M \times 3M$ symmetric matrix, as expected for a linear elastic element. The integrations indicated in Equation 9.117 depend on the specific element type in question. For a four-node, linear tetrahedral element (Section 6.7), all the partial derivatives of the volume coordinates are constants, so the strains are constant—this is the 3-D analogy to a constant strain triangle in two dimensions. In the linear tetrahedral element, the terms of the $[B]$ matrix are constant and the integrations reduce to a constant multiple of element volume.

If the element to be developed is an eight-node brick element, the interpolation functions, Equation 6.69, are such that strains vary linearly and the integrands in Equation 9.117 are not constant. The integrands are polynomials in the spatial variables, however, and therefore amenable to exact integration by Gaussian quadrature in three dimensions. Similarly, for higher-order elements, the integrations required to formulate the stiffness matrix are performed numerically.

The eight-node brick element can be transformed into a generally shaped parallelopiped element using the isoparametric procedure discussed in Section 6.8. If the eight-node element is used as the parent element, the resulting isoparametric element has planar faces and is analogous to the two-dimensional quadrilateral element. If the parent element is of higher-order interpolation functions, an element with general (curved) surfaces results.

Regardless of the specific element type or types used in a three-dimensional finite element analysis, the procedure for assembling the global equilibrium equations is the same as discussed several times, so we do not belabor the point here. As in previous developments, the assembled global equations are of the form

$$[K]\{\Delta\} = \{F\} \tag{9.118}$$

with $[K]$ representing the assembled global stiffness matrix, $\{\Delta\}$ representing the column matrix of global displacements, and $\{F\}$ representing the column matrix of applied nodal forces. The nodal forces may include directly applied external forces at nodes, the work-equivalent nodal forces corresponding to body forces and forces arising from applied pressure on element faces.

9.7 STRAIN AND STRESS COMPUTATION

Using the stiffness method espoused in this text, the solution phase of a finite element analysis results in the computation of unknown nodal displacements as well as reaction forces at constrained nodes. Computation of strain components, then stress components, is a secondary (postprocessing) phase of the analysis. Once the displacements are known, the strain components (at each node in the model) are readily computed using Equation 9.104, which, given the discretization in the finite element context, becomes

$$\{\varepsilon\} = [L]\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [L][N_3]\{\delta\} = [B]\{\delta\} \tag{9.119}$$

It must be emphasized that Equation 9.119 represents the calculation of strain components for an *individual element* and must be carried out for *every* element in the finite element model. However, the computation is straightforward, since the $[B]$ matrix has been computed for each element to determine the element stiffness matrix, hence the element contributions to the global stiffness matrix.

Similarly, element stress components are computed as

$$\{\sigma\} = [D][B]\{\delta\} \tag{9.120}$$

and the material property matrix $[D]$ depends on the state of stress, as previously discussed. Equations 9.119 and 9.120 are general in the sense that the equations are valid for any state of stress if the strain-displacement matrix $[B]$ and the material property matrix $[D]$ are properly defined for a particular state of stress. (In this context, recall that we consider only linearly elastic deformation in this text.)

The element strain and stress components, as computed, are expressed in the element coordinate system. In general, for the elements commonly used in stress analysis, the coordinate system for each element is the same as the global coordinate system. It is a fact of human nature, especially of engineers, that we select the simplest frame in which to describe a particular occurrence or event. This is a way of saying that we tend to choose a coordinate system for convenience and that convenience is most often related to the geometry of the problem at hand. The selected coordinate system seldom, if ever, corresponds to maximum loading conditions. Specifically, if we consider the element stress calculation represented by Equation 9.120, the stress components are referred, and calculated with reference, to a specified Cartesian coordinate system. To determine the critical loading on any model, we must apply one of the so-called failure theories. As we limit the discussion to linearly elastic behavior, the "failure" in our context is yielding of the material. There are several commonly accepted failure theories for yielding in a general state of stress. The two most commonly applied are the *maximum shear stress theory* and the *distortion energy theory.* We discuss each of these briefly. In a general, three-dimensional state of stress, the *principal stresses* σ_1, σ_2, and σ_3 are given by the roots of the cubic equation represented by the determinant [2]

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0 \tag{9.121}$$

Customarily, the principal stresses are ordered so that $\sigma_1 > \sigma_2 > \sigma_3$. Via the usual convention, a positive normal stress corresponds to tension, while a negative normal stress is compressive. So, while σ_3 is algebraically the smallest of the three principal stresses, it may represent a compressive stress having significantly large magnitude. Also recall that the principal stresses occur on mutually orthogonal planes (the *principal planes*) and the shear stress components on those planes are zero.

Having computed the principal stress components, the maximum shear stress is

$$\tau_{\max} = \text{largest of}\left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_1 - \sigma_3|}{2}, \frac{|\sigma_2 - \sigma_3|}{2}\right) \tag{9.122}$$

The three shear stress components in Equation 9.122 are known to occur on planes oriented 45° from the principal planes.

The maximum shear stress theory (MSST) holds that failure (yielding) in a general state of stress occurs when the maximum shear stress as given by Equation 9.122 equals or exceeds the maximum shear stress occurring in a uniaxial tension test at yielding. It is quite easy to show that the maximum shear stress in a tensile test at yielding has value equal to one-half the tensile yield strength of the material. Hence, the failure value in the MSST is $\tau_{\max} = S_y/2 = S_{ys}$. In this notation, S_y is tensile yield strength and S_{ys} represents yield strength in shear.

The distortion energy theory (DET) is based on the strain energy stored in a material under a given state of stress. The theory holds that a uniform tensile or compressive state of stress (also known as *hydrostatic* stress) does not cause distortion and, hence, does not contribute to yielding. If the principal stresses have been computed, total elastic strain energy is given by

$$\begin{aligned} U_e &= \frac{1}{2}\iiint\limits_V (\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)\,\mathrm{d}V \\ &= \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)\right]V \end{aligned} \tag{9.123}$$

To arrive at distortion energy, the average (hydrostatic) stress is defined as

$$\sigma_{\text{av}} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{9.124}$$

and the corresponding strain energy is

$$U_{\text{hyd}} = \frac{3\sigma_{\text{av}}^2}{2E}(1 - 2\nu)V \tag{9.125}$$

The distortion energy is then defined as

$$U_d = U_e - U_{\text{hyd}} \tag{9.126}$$

After a considerable amount of algebraic manipulation, the distortion energy in terms of the principal stress components is found to be given by

$$U_d = \frac{1+\nu}{3E}\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}{2}\right]^{1/2} V \tag{9.127}$$

The DET states that failure (yielding) occurs in a general state of stress when the distortion energy per unit volume equals or exceeds the distortion energy per unit volume occurring in a uniaxial tension test at yielding. It is relatively easy to show (see Problem 9.20) that, at yielding in a tensile test, the distortion energy is given by

$$U_d = \frac{1+\nu}{3E}S_y^2 V \tag{9.128}$$

and, as before, we use S_y to denote the tensile yield strength. Hence, Equations 9.127 and 9.128 give the failure (yielding) criterion for the DET as

$$\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}{2}\right]^{1/2} \geq S_y \tag{9.129}$$

The DET as described in Equation 9.129 leads to the concept of an *equivalent stress* (known historically as the *Von Mises stress*) defined as

$$\sigma_e = \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}{2}\right]^{1/2} \tag{9.130}$$

and failure (yielding) can then be equivalently defined as

$$\sigma_e \geq S_y \tag{9.131}$$

Even though we do not present the algebraic details here, the DET can be shown to be equivalent to another elastic failure theory, known as the *octahedral shear stress theory* (OSST). For all practical purposes, the OSST holds that yielding occurs when the maximum shear stress exceeds $0.577S_y$. In comparison to the MSST, the OSST gives the material more "credit" for strength in shear.

Why do we go into detail on these failure theories in the context of finite element analysis? As noted previously, strain and stress components are calculated in the specified coordinate system. The coordinate system seldom is such that maximum stress conditions are automatically obtained. Here is the point: Essentially every finite element software package not only computes strain and stress components in the global and element coordinate systems but also principal stresses and the equivalent (Von Mises) stress for every element. In deciding whether a design is acceptable (and this is why we use FEA, isn't it?), we must examine the propensity to failure. The examination of stress data is the responsibility of the user of FEA software. The software does *not* produce results that indicate failure unless the analyst carefully considers the data in terms of specific failure criteria.

Among the stress- and strain-related items generally available as a result of solution are the computed stresses (in the specified coordinate system), the principal stresses, the equivalent stress, the principal strains, and strain energy. With the exception of strain energy, the stress data are available on either a nodal or element basis. The distinction is significant, and the analyst must be acutely aware of the distinction. Since strain components (therefore, stress components) are not in general continuous across element boundaries, nodal stresses are computed as average values based on all elements connected to a specific node. On the other hand, element stresses represent values computed at the element centroid. Hence, element stress data are more accurate and should be used in making engineering judgments. To illustrate, we present some of the stress data obtained in the solution of Example 9.4 based on two-dimensional, four-node quadrilateral elements. In the model, node 107 (selected randomly) is common to four elements. Table 9.1 lists the stresses computed at this node in terms of the four connected elements. The values are obtained by computing the nodal stresses for each of the four elements independently, then extracting the values

Table 9.1 Stress Values (psi) Computed at Node 107 of Example 9.4

	σ_x	σ_y	τ_{xy}
Element 1	2049.3	187.36	118.4
Element 2	2149.4	315.59	91.89
Element 12	1987.3	322.72	204.13
Element 99	1853.8	186.88	378.36
Average	2009.8	253.14	198.19

Table 9.2 Element Stress Components (psi) for Four Elements Sharing a Common Node in Example 9.4

	σ_x	σ_y	τ_{xy}	σ_e
Element 1	2553.5	209.71	179.87	2475.8
Element 2	1922.7	351.69	43.55	1774.8
Element 12	1827.5	264.42	154.44	1731.5
Element 99	2189.0	249.14	480.57	2236.4

for the common node. The last row of the table lists the average values of the three stress components at the common node. Clearly, the nodal stresses are not continuous from element to element at the common node. As previously discussed, the magnitudes of the discontinuities should decrease as the element mesh is refined.

In contrast, the *element* stress components for the same four elements are shown in Table 9.2. The values listed in the table are computed at the element centroid and include the equivalent (Von Mises) stress as defined previously. While not included in the table, the principal stress components are also available from the solution. In general, the element stresses should be used in results evaluation, especially in terms of application of failure theories.

9.8 PRACTICAL CONSIDERATIONS

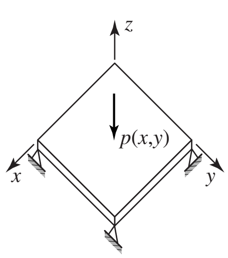


Figure 9.12 Example of a thin plate subjected to bending.

Probably the most critical step in application of the finite element method is the choice of element type for a given problem. The solid elements discussed in this chapter are among the simplest elements available for use in stress analysis. Many more element types are available to the finite element analyst. (One commercial software system has no fewer than 141 element types.) The differences in elements for stress analysis fall into three categories: (1) number of nodes, hence, polynomial order of interpolation functions; (2) type of material behavior (elastic, plastic, thermal stress, for example); and (3) loading and geometry of the structure to be modeled (plane stress, plane strain, axisymmetric, general three dimensional, bending, torsion).

As an example, consider Figure 9.12, which shows a flat plate supported at the corners and loaded by a pressure distribution $p(x, y)$ acting in the negative z direction. The primary mode of deformation of the plate is bending in the z direction. To adequately describe the behavior, a finite element used to model the plate must be such that continuity of slope in both xz and yz planes is ensured. Therefore, a three-dimensional solid element as described in Section 9.8 would not be appropriate as only the displacement components are included as nodal variables. Instead, an element that includes partial derivatives representing the slopes must be included as nodal variables. Plate elements have been developed on the basis of the theory of thin plates (usually only covered in graduate programs) in which

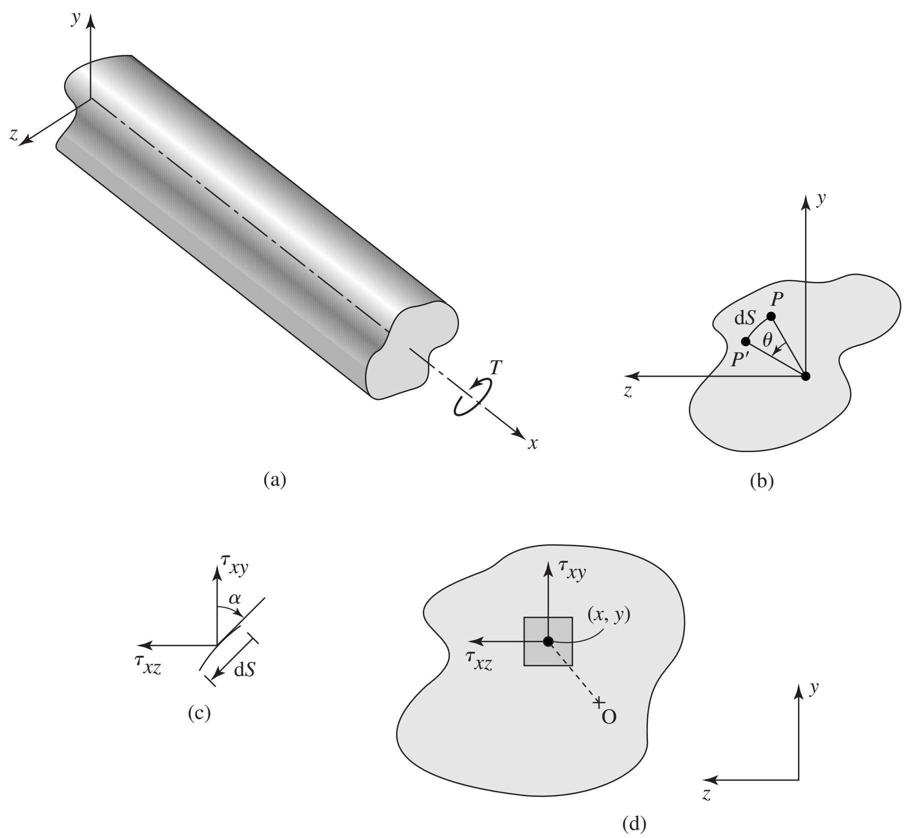


Figure 9.13
(a) A general, noncircular section in torsion. (b) Motion of a point from P to P' as a result of cross-section rotation. (c) A differential element at the surface of a torsion member. (d) A differential element showing the contribution of shear stress to torque.

the bending deflection is governed by a fourth-order partial differential equation. The simplest such element is a four-node element using cubic interpolation functions and having 4 degrees of freedom (displacement, two slopes, and a mixed second derivative) at each node [4]. A similar situation exists with shell (thin curved plate) structures. Specialized elements are required (and available) for structural analysis of shell structures. The major point here is that a breadth of knowledge and experience is required for a finite element analyst to become truly proficient at selecting the correct element type(s) for a finite element model and, subsequently interpreting the results of the analysis.

Once the element type has been selected, the task becomes that of defining the model geometry as a mesh of finite elements. In its most rudimentary form, this task involves defining the coordinate location of every node in the model (note that, by default, the nodes define the geometry) followed by definition of all elements in terms of nodes. Many years ago, in the early development of the finite element method, the tasks of node and element definition were labor intensive, as the definitions required use of the specific language statements of a particular finite element software system. The tasks were laborious, to say the least, and prone to error. With currently technology, especially graphical user interfaces and portability of computer-aided design (CAD) databases, these tasks have been greatly simplified. It is now possible, with many FEA programs, to “import” the geometry of a component, structure, or assembly directly from a CAD system, so that geometry does not need to be defined. The finite element software can then automatically create a mesh (*automeshing*) of finite elements to represent the geometry. The advantages of this capability include (1) the finite element analyst need not redefine the geometry; consequently, (2) the designer’s intent is not changed inadvertently; and (3) the finite element analyst is relieved of the burden of specifying the details of the node and element definitions. The major disadvantage is that the analyst is not in *direct* control of the meshing operation.

The word direct is emphasized. In automeshing, the software user has some control over the meshing process. There are two general types of automeshing software, generally referred to as *free meshing* and *mapped meshing*. In free meshing, the user specifies a general, qualitative mesh description, ranging from coarse to fine, with 10 or more gradations between the extremes. The software then generates the mesh accordingly. In mapped meshing, the user specifies quantitative information regarding node spacing, hence, element size, and the software uses the prescribed information to generate nodes and elements. In either method, the software user has some degree of control over the element mesh.

A very important aspect of meshing a model with elements is to ensure that, in regions of geometric discontinuity, a finer mesh (smaller elements) is defined in the region. This is true in all finite element analyses (structural, thermal, and fluid), because it is known that gradients are higher in such areas and finer meshes are required to adequately describe the physical behavior. In mapped meshing, this is defined by the software user. Fortunately, in free meshing, this aspect is accounted for in the software. As an example, refer back to Example 9.4, in which we examined the stress concentration factor for a hole in a thin plate subjected to tension. The solution was modeled using the free mesh feature of a finite element software system. Figure 9.8b is a coarse mesh as generated by the software. Geometry is defined by four lines and a quarter-circular arc; these, in turn, define a single area of interest. Having specified the element type (in this case a plane stress, elastic, quadrilateral), the area meshing feature is used to generate the elements as shown. It is important to note the relatively fine mesh in the vicinity of the arc representing the hole. This is generated by the software automatically in recognition of the geometry. The mesh-refined models of Figure 9.8c and 9.8d are also generated by the free meshing routine. From each of these cases, we see that, not only does the number of elements increase, but the relative size of the elements in the vicinity of the hole is maintained relative to elements far removed from the discontinuity.

The automeshing capabilities of finite software as briefly described here are extremely important in reducing the burden of defining a finite element model of any geometric situation and should be used to the maximum extent. However, recall that the results of a finite element analysis must be judged by human knowledge of engineering principles. Automated model definition is a nicety of modern finite element software; automated analysis of results is not.

Analysis of results is the postprocessing phase of finite element analysis. Practical models contain hundreds, if not thousands, of elements, and the computed displacements, strains, stresses, and so forth are available for every element. Poring through the data can be a seemingly endless task. Fortunately, finite element software has, as part of the postprocessing phase, routines for sorting the results data in many ways. Of particular importance in stress analysis, the data can be sorted in ascending or descending order of essentially any stress component chosen by the user. Hence, one can readily determine the maximum equivalent stress, for example, and determine the location of that stress by the associated element location. In addition, with modern computer technology, it is possible to produce color-coded stress contour plots of an entire model, to visually observe the stress distribution, the deformed shape, the strain energy distribution, and many other criteria.

9.9 TORSION

Torsion (twisting) of structural members having circular cross sections is a common problem studied in elementary mechanics of materials. (Recall that earlier we developed a finite element for such cases.) A major assumption (and the assumption is quite valid for elastic deformation) in torsion of circular members is that plane sections remain plane after twisting. In the case of torsion of a noncircular cross section, this assumption is not valid and the problem is hence more complicated. A general structural member subjected to torsion is shown in Figure 9.13a. The member is subjected to torque T acting about the x axis, and it is assumed that the cross section is uniform along the length. An arbitrary point located on a cross section at position x is shown in Figure 9.13b. If the cross section twists through angle θ, the point moves through arc ds and the displacement components in the y and z directions are

$$\begin{aligned} v &= -z\theta \\ w &= y\theta \end{aligned} \tag{9.132}$$

respectively. Since the angle of twist varies along the length of the member, we conclude that the displacement components of Equation 9.132 are described by

$$v = v(x, z) \qquad w = w(x, y) \tag{9.133}$$

Owing to the noncircular cross section, plane sections do not remain plane; instead, there is warping, hence displacement, in the x direction described by

$$u = u(y, z) \tag{9.134}$$

Applying the definitions of the normal strain components to Equation 9.133 and 9.134, we find

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = 0 \\ \varepsilon_y &= \frac{\partial v}{\partial y} = 0 \\ \varepsilon_z &= \frac{\partial w}{\partial z} = 0 \end{aligned} \tag{9.135}$$

and it follows that the normal stress components are $\sigma_x = \sigma_y = \sigma_z = 0$. To compute the shear strain components, we introduce the angle of twist *per unit length* ϕ such that the rotation of any cross section can be expressed as $\theta = \phi x$. The displacement components are then expressed as

$$u = u(y,\ z) \qquad v = -\phi x z \qquad w = \phi x y \tag{9.136}$$

and the shear strain components are

$$\begin{aligned} \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} - \phi z \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} + \phi y \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \end{aligned} \tag{9.137}$$

It follows from the stress-strain relations that the only nonzero stress components are τ_{xy} and τ_{xz} and the only equilibrium equation (Appendix B) not identically satisfied becomes

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{9.138}$$

We now hypothesize the existence of a scalar function $\psi(y, z)$ such that

$$\tau_{xy} = \frac{\partial \psi}{\partial z} \qquad \tau_{xz} = -\frac{\partial \psi}{\partial y} \tag{9.139}$$

In the context of the torsion problem, scalar function ψ is known as *Prandtl's stress function* and is generally analogous to the stream function and potential function introduced in Chapter 8 for ideal fluid flow. If the relations of Equation 9.139 are substituted into Equation 9.138, we find that the equilibrium condition is automatically satisfied. To discover the governing equation for the stress function, we compute the stress components as

$$\tau_{xy} = G\gamma_{xy} = G\left(\frac{\partial u}{\partial y} - \phi z\right)$$
$$\tau_{xz} = G\gamma_{yz} = G\left(\frac{\partial u}{\partial z} + \phi y\right) \tag{9.140}$$

and note that

$$\frac{\partial \tau_{xy}}{\partial z} = G\left(\frac{\partial^2 u}{\partial y\, \partial z} - \phi\right)$$
$$\frac{\partial \tau_{xz}}{\partial y} = G\left(\frac{\partial^2 u}{\partial y\, \partial z} + \phi\right) \tag{9.141}$$

Combining the last two equations results in

$$\frac{\partial \tau_{xy}}{\partial z} - \frac{\partial \tau_{xz}}{\partial y} = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -2G\phi \tag{9.142}$$

as the governing equation for Prandtl's stress function. As with the fluid formulations of Chapter 8, note the analogy of Equation 9.142 with the case of heat conduction. Here the term $2G\phi$ is analogous to internal heat generation Q.

9.9.1 Boundary Condition

At the outside surface of the torsion member, no stress acts normal to the surface, so the resultant of the shear stress components must be tangent to the surface. This is illustrated in Figure 9.13c showing a differential element dS of the surface (with the positive sense defined by the right-hand rule). For the normal stress to be zero, we must have

$$\tau_{xy} \sin\alpha - \tau_{xz} \cos\alpha = 0 \tag{9.143}$$

or

$$\tau_{xy} \frac{\mathrm{d}z}{\mathrm{d}s} - \tau_{xz} \frac{\mathrm{d}y}{\mathrm{d}s} = 0 \tag{9.144}$$

Substituting the stress function relations, we obtain

$$\frac{\partial \psi}{\mathrm{d}z} \frac{\mathrm{d}z}{\mathrm{d}s} + \frac{\mathrm{d}\psi}{\mathrm{d}y} \frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\mathrm{d}\psi}{\mathrm{d}s} = 0 \tag{9.145}$$

which shows that the value of the stress function is constant on the surface. The value is arbitrary and most often taken to be zero.

9.9.2 Torque

The stress function formulation of the torsion problem as given previously does not explicitly include the applied torque. To obtain an expression relating the applied torque and the stress function, we must consider the moment equilibrium condition. Referring to the differential element of a cross section shown in Figure 9.13d, the differential torque corresponding to the shear stresses acting on the element is

$$\mathrm{d}T = (y\tau_{xz} - z\tau_{xy})\,\mathrm{d}A \tag{9.146}$$

and the total torque is computed as

$$T = \iint\limits_A (y\tau_{xz} - z\tau_{xy})\,\mathrm{d}A = -\iint\limits_A \left(y\frac{\partial\psi}{\partial y} + z\frac{\partial\psi}{\mathrm{d}z}\right)\mathrm{d}y\,\mathrm{d}z = 2\iint\limits_A \psi\,\mathrm{d}A \tag{9.147}$$

The final result in the last equation is obtained by integrating by parts and noting the condition $\psi = 0$ on the surface.

9.9.3 Finite Element Formulation

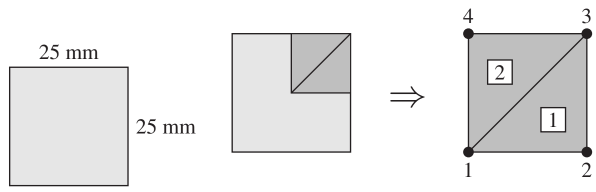
Since the governing equation for the stress function is analogous to the heat conduction equation, it is not necessary to repeat the details of element formulation. Instead, we reiterate the analogies and point out one very distinct difference in how a finite element analysis of the torsion problem is conducted when the stress function is used. First, note that the stress function is discretized as

$$\psi\{y, z\} = \sum_i^M N_i(y, z)\psi_i \tag{9.148}$$

so that the finite element computations result in nodal values analogous to nodal temperatures (with the conductivity values set to unity). Second, the torsion term $2G\phi$ is analogous to internal heat generation Q. However, the angle of twist per unit length ϕ is actually the unknown we wish to compute in the first place. Preferably, in such a problem, we specify the geometry, material properties, and applied torque, then compute the angle of twist per unit length as well as stress values. However, the formulation here is such that we must specify a value for angle of twist per unit length, compute the nodal values of the stress function, then obtain the torque by summing the contributions of all elements to Equation 9.147. Since the governing equation is linear, the angle of twist per unit length and the computed torque can be scaled in ratio as required. The procedure is illustrated in the following example.

EXAMPLE 9.7

Figure 9.14a shows a shaft having a square cross section with 50-mm sides. The material has shear modulus 80 Gpa. Shaft length is 1 m. The shaft is fixed at one end and subjected to torque T at the other end. Determine the total angle of twist if the applied torque is 100 N-m.



(a) (b)

Figure 9.14 Finite element model of Example 9.7.

■ Solution

Observing the symmetry conditions, we model one-fourth of the cross section using three-node linear triangular elements, as in Figure 9.14b. For simplicity of illustration, we use only two elements and note that, at nodes 2, 3, and 4, the value of the stress function is specified as zero, since these nodes are on the surface. Also note that the planes of symmetry are such that the partial derivatives of the stress function across those planes are zero. These conditions correspond to zero normal heat flux (perfect insulation) in a conduction problem.

The element stiffness matrices are given by

$$\left[k^{(e)}\right] = \iint\limits_{A} \left(\left[\frac{\partial N}{\partial y}\right]^T \left[\frac{\partial N}{\partial y}\right] + \left[\frac{\partial N}{\partial z}\right]^T \left[\frac{\partial N}{\partial z}\right] \right) \mathrm{d}A$$

and element nodal forces are

$$\left\{f^{(e)}\right\} = \iint\limits_{A} 2G\phi\,[N]^T\,\mathrm{d}A$$

(Note the use of y, z coordinates in accord with the coordinate system used in the preceding developments.)

These relations are obtained by analogy with Equation 7.35 for heat conduction. The interpolation functions are as defined in Equation 9.28.

Element 1

$$N_1 = \frac{1}{2A}(625 - 25y) \qquad \frac{\partial N_1}{\partial y} = -\frac{25}{2A} \qquad \frac{\partial N_1}{\partial yz} = 0$$

$$N_2 = \frac{1}{2A}(25y - 25z) \qquad \frac{\partial N_2}{\partial y} = \frac{25}{2A} \qquad \frac{\partial N_2}{\partial z} = -\frac{25}{2A}$$

$$N_3 = \frac{1}{2A}(25z) \qquad \frac{\partial N_3}{\partial y} = 0 \qquad \frac{\partial N_3}{\partial z} = \frac{25}{2A}$$

Since the partial derivatives are all constant, the stiffness matrix is

$$\left[k^{(1)}\right] = \frac{1}{4A}\begin{Bmatrix} -25 \\ 25 \\ 0 \end{Bmatrix}[-25 \quad 25 \quad 0] + \frac{1}{4A}\begin{Bmatrix} 0 \\ -25 \\ 25 \end{Bmatrix}[0 \quad -25 \quad 25]$$

or

$$[k^{(1)}] = \frac{1}{1250}\begin{bmatrix} 625 & -625 & 0 \\ -625 & 625 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{1250}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 625 & -625 \\ 0 & -625 & 625 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

The element nodal forces are readily shown to be given by

$$\{f^{(1)}\} = \frac{2G\phi A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

which we leave in this general form for the time being.

Element 2

$$N_1 = \frac{1}{2A}(625 - 25z) \qquad \frac{\partial N_1}{\partial y} = 0 \qquad \frac{\partial N_1}{\partial z} = -\frac{25}{2A}$$

$$N_2 = \frac{1}{2A}(25y) \qquad \frac{\partial N_2}{\partial y} = \frac{25}{2A} \qquad \frac{\partial N_2}{\partial z} = 0$$

$$N_3 = \frac{1}{2A}(25z - 25y) \qquad \frac{\partial N_3}{\partial y} = -\frac{25}{2A} \qquad \frac{\partial N_3}{\partial z} = \frac{25}{2A}$$

$$[k^{(2)}] = \frac{1}{4A}\begin{Bmatrix} 0 \\ 25 \\ -25 \end{Bmatrix}[0 \quad 25 \quad -25] + \frac{1}{4A}\begin{Bmatrix} -25 \\ 0 \\ 25 \end{Bmatrix}[-25 \quad 0 \quad 25]$$

$$[k^{(2)}] = \frac{1}{1250}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 625 & -625 \\ 0 & -625 & 625 \end{bmatrix} + \frac{1}{1250}\begin{bmatrix} 625 & 0 & -625 \\ 0 & 0 & 0 \\ -625 & 0 & 625 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0.5 & -0.5 \\ -0.5 & -0.5 & 1.0 \end{bmatrix}$$

For element 2, the nodal forces are also given by

$$\{f^{(2)}\} = \frac{2G\phi A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Noting the element to global nodal correspondences, the assembled system equations are

$$\begin{bmatrix} 1 & -0.5 & 0 & -0.5 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ -0.5 & 0 & -0.5 & 1 \end{bmatrix}\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{Bmatrix} = \frac{2G\phi A}{3}\begin{Bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{Bmatrix}$$

Since nodes 2, 3, and 4 are on the outside surface, we set $\psi_2 = \psi_3 = \psi_4 = 0$ to obtain the solution

$$\psi_1 = \frac{4G\phi A}{3}$$

We still have not addressed the problem that the angle of twist per unit length is unknown and continue to ignore that problem temporarily, since for this very simple two-element model, we can continue with the hand solution. The torque is given by Equation 9.149 as

$$T = 2\iint_A \psi\,\mathrm{d}A$$

but the integration is over the entire cross section. Hence, we must sum the contribution from each element and, in this case, since we applied the symmetry conditions to reduce the model to one-fourth size, multiply the result by 4. For each element, the torque contribution is

$$T^{(e)} = 2\iint_A \left[N^{(e)}\right]\mathrm{d}A\left\{\psi^{(e)}\right\}$$

and for the linear triangular element this simply becomes

$$T^{(e)} = \frac{2A}{3}\left(\psi_1^{(e)} + \psi_2^{(e)} + \psi_3^{(e)}\right)$$

Accounting for the use of symmetry, the total torque indicated by our two-element solution is

$$T = 4\frac{2A}{3}\left(\psi_1^{(1)} + \psi_2^{(2)}\right) = \frac{64}{9}G\phi A^2$$

or

$$\frac{T}{\phi} = \frac{64}{9}GA^2$$

Now we address the problem of unknown angle of twist per unit length. Noting, in the last equation, the ratio is constant for specified shear modulus and cross-sectional area, we could simply specify an arbitrary value of ϕ, follow the solution procedure to compute the corresponding torque, compute the ratio, and scale the result as needed. If, for example, we had assumed $\phi = 10^{-6}$ rad/mm, our result would be

$$T = \frac{64}{9}(80)(10^3)(10^{-6})(312.5)^2 = 55555.6 \text{ N·mm} \Rightarrow 55.6 \text{ N·m}$$

Thus, to answer the original question, we compute the angle of twist per unit length corresponding to the specified torque as

$$\phi = \frac{100}{55.6}(10^{-6}) \approx 1.8(10^{-6}) \text{ rad/mm}$$

and the total angle of twist would be

$$\theta = \phi L = 1.8(10^{-6})(1000) = 1.8(10^{-3}) \text{ rad}$$

or about 0.1 degree. The exact solution [5] for this problem shows the angle of twist per unit length to be $\phi = 1.42(10^{-6})$ rad/mm. Hence, our very simple model is in error by about 27 percent.

9.10 SUMMARY

In this chapter, we present the development of the most basic finite elements used in stress analysis in solid mechanics. As the finite element method was originally developed for stress analysis, the range of element and problem types that can be analyzed by the method are very large. Our description of the basic concepts is intended to give the reader insight on the general procedures used to develop element equations and understand the ramifications on element, hence, model, formulation for various states of stress. As mentioned in the context of plate bending in Section 9.8, finite element analysis involves many advanced topics in engineering not generally covered in an undergraduate program. The interested reader is referred to the many advanced-level texts on the finite element method for further study. The intent here is to introduce the basic concepts and generate interest in learning more of the subject.

REFERENCES

1. Kreyszig, E. *Advanced Engineering Mathematics,* 5th ed. New York: John Wiley and Sons, 1983.

2. Shigley, J. E. *Mechanical Engineering Design,* 6th ed. New York: McGraw-Hill, 1998.

3. Cowper, G. R. "Gaussian Quadrature Formulas for Triangles." *International Journal for Numerical Methods in Engineering* 7 (1973).

4. Huebner, K. H., and E. A. Thornton. *The Finite Element Method for Engineers,* 2nd ed. New York: John Wiley and Sons, 1982.

5. Timoshenko, S. P., and J. N. Goodier. *Theory of Elasticity,* New York: McGraw-Hill, 1970.

PROBLEMS

9.1 Use the general stress-strain relations from Appendix B and the assumptions of plane stress to derive Equations 9.2.

9.2 Let $\{z\}$ be the $N \times 1$ column matrix $[z_1 \ \ z_2 \ \ z_3 \ \ \cdots \ \ z_N]^T$ and let $[A]$ be an $N \times N$ real-valued matrix. Show that the matrix product $\{z\}^T[A]\{z\}$ always results in a scalar, quadratic function of the components z_i.

9.3 Beginning with the general elastic stress-strain relations, derive Equation 9.50 for the conditions of plane strain.

9.4 Determine the strain-displacement matrix $[B]$ for a three-node triangular element in plane strain.

9.5 Determine the strain-displacement matrix $[B]$ for a four-node rectangular element in plane stress.

9.6 Using the interpolation functions given in Equation 9.28, determine the explicit expression for the strain energy in a three-node triangular element in plane stress.

9.7 The constant strain triangular element shown in Figure P9.7 is subjected to a uniformly distributed pressure as shown. Determine the equivalent nodal forces.

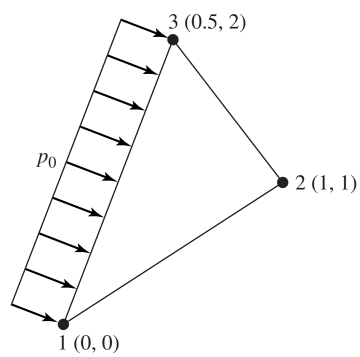


Figure P9.7

9.8 The constant strain triangular element shown in Figure P9.8 is subjected to the linearly varying pressure as shown and a body force from gravity ($g = 386.4$ in./s^2) in the negative y direction. Determine the equivalent nodal forces.

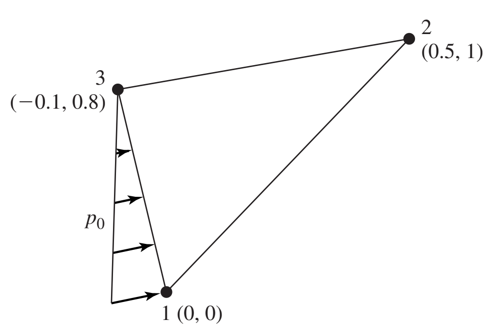


Figure P9.8

9.9 The element of Figure P9.7 is of a material for which the modulus of elasticity is $E = 15 \times 10^6$ psi and Poisson's ratio is $\nu = 0.3$. Determine the element stiffness matrix if the element is subjected to plane stress.

9.10 Repeat Problem 9.9 for plane strain.

9.11 Repeat Problem 9.9 for an axisymmetric element.

9.12 The overall loading of the element in Problem 9.7 is such that the nodal displacements are $u_1 = 0.003$ in., $v_1 = 0$, $u_2 = 0.001$ in., $v_2 = 0.0005$ in., $u_3 = 0.0015$ in., $v_3 = 0$. Calculate the element strain, stress, and strain energy assuming plane stress conditions.

9.13 Repeat Problem 9.13 for plane strain conditions.

9.14 A thin plate of unit thickness is supported and loaded as shown in Figure P9.14. The material is steel, for which $E = 30 \times 10^6$ and $\nu = 0.3$. Using the four constant strain triangular elements shown by the dashed lines, compute the deflection of point A. Compare the total strain energy with the work of the external force system.

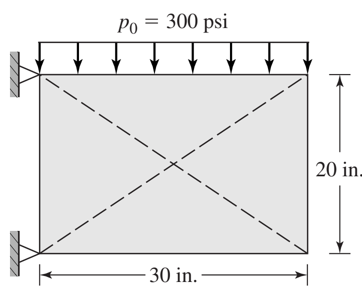


Figure P9.14

9.15 Integrate Equation 9.65 by the Gaussian numerical procedure to verify Equation 9.66.

9.16 A three-node triangular element having nodal coordinates, shown in Figure P9.16, is to be used as an axisymmetric element. The material properties are $E = 82$ GPa and $\nu = 0.3$. The dimensions are in millimeters. Calculate the element stiffness matrix using both the exact definition of Equation 9.99 and the centroidal approximation of Equation 9.100. Are the results significantly different?

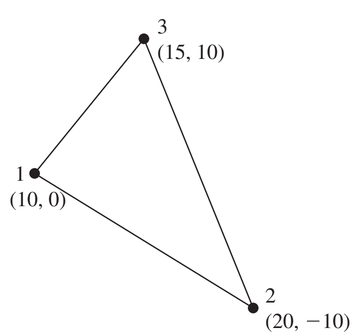


Figure P9.16

9.17 The axisymmetric element in Figure P9.16 is subjected to a uniform, normal pressure p_0 acting on the surface defined by nodes 1 and 3. Compute the equivalent nodal forces.

9.18 The axisymmetric element in Figure P9.16 is part of a body rotating about the z axis at a constant rate of 3600 revolutions per minute. Determine the corresponding nodal forces.

9.19 Consider the higher-order three-dimensional element shown in Figure P9.19, which is assumed to be subjected to a general state of stress. The element has 20 nodes, but all nodes are not shown for clarity.

a. What is the order of the polynomial used for interpolation functions?
b. How will the strains (therefore, stresses) vary with position in the element?
c. What is the size of the stiffness matrix?
d. What advantages and disadvantages are apparent in using this element in comparison to an eight-node brick element?

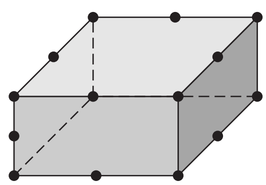


Figure P9.19

9.20 Show that in a uniaxial tension test the distortion energy at yielding is given by Equation 9.128.

9.21 A finite element analysis of a certain component yields the maximum principal stresses $\sigma_1 = 200$ MPa, $\sigma_2 = 0$, $\sigma_3 = -90$ MPa. If the tensile strength of the material is 270 MPa, is yielding indicated according to the distortion energy theory? If not, what is the "safety factor" (ratio of yield strength to equivalent stress)?

9.22 Repeat Problem 9.21 if the applicable failure theory is the maximum shear stress theory.

9.23 The torsion problem as developed in Section 9.9 has a governing equation analogous to that of two-dimensional heat conduction. The stress function is analogous to temperature, and the angle of twist per unit length term ($2G\phi$) is analogous to internal heat generation.

a. What heat transfer quantities are analogous to the shear stress components in the torsion problem?
b. If one solved a torsion problem using finite element software for two-dimensional heat transfer, how would the torque be computed?

9.24 The torsion problem as developed in Section 9.9 is two-dimensional when posed in terms of the Prandtl stress function. Could three-dimensional elastic solid elements (such as the eight-node brick element) be used to model the torsion problem? If yes, how would a pure torsional loading be applied?

9.25 Figure P9.25 shows the cross section of a hexagonal shaft used in a quick-change power transmission coupling. The shear modulus of the material is 12×10^6 psi

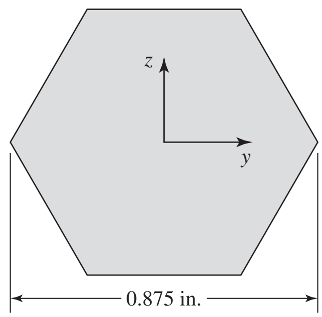


Figure P9.25

and the shaft length is 12 in. Determine the total angle of twist when the shaft is subjected to a net torque of 250 ft-lb. Use linear triangular elements and take advantage of all appropriate symmetry conditions.